

Graphics 2009/2010

T1

Midterm exam

Mon, Sept 28, 2009, 08:30–10:30

- **Do not open this exam until instructed to do so.**
- **Read the instructions on this page carefully.**

- You may write your answers in English, Dutch, or German.
Use a pen, not a pencil. Avoid usage of the color red.
- Write down your name and student number on every paper you want to turn in.
Additional paper is provided by us. You are not allowed to use your own paper.
- You may not use books, notes, or any electronic equipment
(including your cellphone, even if you just want to use it as a clock).

- The exam should be doable in less than 1.5 hours. You have max. 2 hours to work on the questions.
If you finish early, you may hand in your work and leave, except for the first half hour of the exam.
- When you hand in your work, have your student ID ready for inspection and write your name and student number on the list of participants.

- The exam consists of 4 problems printed on 5 pages (including this one).
It is your responsibility to check if you have a complete printout.
If you have the impression that anything is missing, let us know.
- The maximum number of points you can score is 18.
You need at least 16 points to get the best possible grade.

Good luck!

Problem 1: Vectors

Subproblem 1.1 [1 pt] Assume s is a scalar value, and \vec{v} and \vec{w} are two vectors in \mathbb{R}^3 . “ \times ” denotes the cross product of two vectors and “ \cdot ” denotes the scalar product (or inner product or dot product).

Which of the following answers **(i)-(iii)** is correct (shortly explain your answer)?

1. The result of $(\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})$ is **(i)** a scalar value, **(ii)** a vector in \mathbb{R}^3 , or **(iii)** undefined?
2. The result of $(s \times \vec{w}) \cdot (s \times \vec{w})$ is **(i)** a scalar value, **(ii)** a vector in \mathbb{R}^3 , or **(iii)** undefined?
3. The result of $s\vec{v} + \vec{v} \times \vec{w}$ is **(i)** a scalar value, **(ii)** a vector in \mathbb{R}^3 , or **(iii)** undefined?
4. The result of $s\vec{w} + \vec{v} \cdot \vec{w}$ is **(i)** a scalar value, **(ii)** a vector in \mathbb{R}^3 , or **(iii)** undefined?

Subproblem 1.2 [1 pt] Assume that vector $\vec{v} \in \mathbb{R}^3$ is a scalar multiple of a vector \vec{w} , i.e. $\vec{v} = \lambda\vec{w}$ with some $\lambda \neq 0$. Prove that the length of vector \vec{v} is λ times the length of vector \vec{w} .

Subproblem 1.3 [1 pt] Assume that \vec{v} and \vec{w} are two unit vectors in \mathbb{R}^3 .

(a) What do we know about \vec{v} and \vec{w} if their scalar product is zero, i.e. if $\vec{v} \cdot \vec{w} = 0$?
Shortly explain your answer.

(b) What do we know about \vec{v} and \vec{w} if their scalar product is one, i.e. if $\vec{v} \cdot \vec{w} = 1$?
Shortly explain your answer.

Problem 2: Basic geometric entities

Subproblem 2.1 [1 pt] Assume the following two points in \mathbb{R}^3 :

$$\vec{p}_0 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \vec{p}_1 = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$$

Give a parametric representation of a line in \mathbb{R}^3 that goes through these two points.

(Note: write down what you are doing and shortly explain the single steps, so we see that you understand what you are doing and also that we can give you some credits even if your solution is wrong.)

Subproblem 2.2 [1.5 pt] Assume the following three points in \mathbb{R}^3 :

$$\vec{q}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{q}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \vec{q}_2 = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

Give an implicit representation of a plane in \mathbb{R}^3 that goes through these three points.

(Note: write down what you are doing and shortly explain the single steps, so we see that you understand what you are doing and also that we can give you some credits even if your solution is wrong.)

Subproblem 2.3 [1 pt] Calculate the intersection of the plane and the line that you created in the preceding two subproblems. What is the geometric interpretation of your solution?

What other options do exist when you intersect a line with a plane in 3D (indicate the number of possible solutions and the geometric interpretation of it)?

Subproblem 2.4 [1.5 pt] One characteristic of a parametric equation is that it is controlled by less parameters than dimensions in the space. For example, for the parametric equation of a sphere in 3D, we have two controlling parameters (or in other words: we are able to uniquely describe and address each 3-dimensional point on the sphere by specifying just two parameters).

(a) What is the controlling parameter in the line equation that you created in the first subproblem? Explain how it can be used to uniquely describe each point on the 2-dimensional line. (Hint: it might help to describe the geometric interpretation of the components of the whole parametric equation first).

(b) Write down the general form of a parametric equation for a circle around the origin in 2D. What is the controlling parameter in this equation? Shortly explain how it can be used to describe every point on the circle. (Hint: it might help to draw such a circle and a vector that points from the origin to a random point on the circle.)

Problem 3: Matrices

Subproblem 3.1 [1 pt] Assume that A and B are two $n \times n$ matrices. (Note: this also means that matrix multiplication between them is defined.)

Are the following statements correct or not? Give a proof of your answer in both cases.

1. $AB = BA$
2. $A(BC) = A(CB)$

Subproblem 3.2 [1.5 pt] Calculate the inverse A^{-1} of the following matrix using Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Note: you have to use Gaussian elimination to solve this problem (or the “forward-backward-step” variation that was mentioned in the lecture), otherwise you will get no credits (even if your result is correct).

Write down each step, so we can understand what you were doing and also can give you some credits even if your solution is wrong.

Subproblem 3.3 [1.5 pt] Assume we have three linear equations - each with three unknown parameters. We can interpret each of these equations as a plane in 3D. Solving the resulting system of linear equations (e.g.) by Gaussian elimination gives us the intersection of these three planes.

What kind of solutions can we get? For each of the possible cases: describe what happens when we solve the linear equation system with Gaussian elimination and discuss *all* possible geometric interpretations.

Hints: if you have problems thinking of all possible geometric interpretations, it might help to first think about what you can get when you intersect two planes, and then what possibilities exist if you calculate the intersection of that result with the remaining third plane.

Subproblem 3.4 [1 pt] One way to calculate determinants is via cofactors. Calculate the cofactor a_{12}^c for the coefficient a_{12} of the following matrix:

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

Note: you do *not* have to calculate the whole determinant, but just the one cofactor for a_{12} . Write down each step, so we can understand what you were doing and also can give you some credits even if your solution is wrong.

Problem 4: Transformations

Subproblem 4.1 [1 pt] A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a linear transformation if it satisfies certain conditions. What are these conditions?

Subproblem 4.2 [1 pt] Prove that scaling in 2D, i.e. multiplication of a vector $\vec{v} = (x, y)$ with the following matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

is a linear transformation. (Note: to do this, you should use what you wrote down in the preceding subproblem)

Subproblem 4.3 [2 pt] In \mathbb{R}^2 , the matrix

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

defines a counterclockwise rotation by the angle ϕ around the origin.

(a) What is the rotation matrix for *clockwise* rotation around the origin in \mathbb{R}^2 ?
Shortly explain how you got your solution.

(b) Give a 3x3 transformation matrix for a similar rotation around the z-axis in \mathbb{R}^3 .
Shortly explain how you got your solution.

Subproblem 4.4 [1 pt] Describe in your own words what happens to a point \vec{p} in \mathbb{R}^3 if you apply the following transformation matrix to it:

$$\begin{pmatrix} -1 & 0 & 0 & x_m \\ 0 & 1 & 0 & y_m \\ 0 & 0 & 1 & z_m \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

How are the values in the last row of this matrix called and why do we need them?