Graphics 2009/2010

T1

Midterm exam

Mon, Sept 28, 2009, 08:30–10:30

SKETCHES AND SOLUTIONS

Note that for most problems, there can be more than one correct solution. Also, the following sketches should not be considered as standard solutions but rather as detailed comments including explanations that may go way beyond what was required to achieve the maximum credits for a particular subproblem. In addition, we've been rather generous in the grading if we realized that there have been obvious misunderstandings with the problem description.

No responsibility is taken for the correctness of the provided information.

Problem 1: Vectors

Subproblem 1.1 [1 pt] Assume s is a scalar value, and $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^3$. "$\times$" denotes the cross product of two vectors and "$\cdot$" denotes the scalar product (or inner product or dot product).

Which of the following answers **(i)-(iii)** is correct (shortly explain your answer)?

1. The result of $(\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})$ is **(i)** a scalar value, **(ii)** a vector in $\mathbb{R}^3$, or **(iii)** undefined?
2. The result of $(s \times \vec{w}) \cdot (s \times \vec{w})$ is **(i)** a scalar value, **(ii)** a vector in $\mathbb{R}^3$, or **(iii)** undefined?
3. The result of $s\vec{v} + \vec{v} \times \vec{w}$ is **(i)** a scalar value, **(ii)** a vector in $\mathbb{R}^3$, or **(iii)** undefined?
4. The result of $s\vec{w} + \vec{v} \cdot \vec{w}$ is **(i)** a scalar value, **(ii)** a vector in $\mathbb{R}^3$, or **(iii)** undefined?

Solution/comments. *In the first case, we take the cross product of two vectors, which is a vector. Then we take the scalar product with another cross product (which is again a vector). The scalar product of two vectors is a scalar value (hence the name* scalar *product). Thus, answer **(i)** is correct.*

*In the second case, we are trying to take the cross product of a scalar value with a vector which is undefined. Hence, the whole equation is undefined and answer **(iii)** is correct.*

*In the third case, we multiply a scalar value with a 3-dimensional vector which gives us another 3-dimensional vector. Since the cross product of two 3-dimensional vectors is again a 3-dimensional vector, the sum is defined and again a 3-dimensional vector. Hence, answer **(ii)** is correct.*

*In the fourth case, we first get the scalar multiple of a vector (which is again a vector) and the scalar product of two vectors (which is a scalar value). Because the sum of a vector with a scalar value is not defined, answer **(iii)** is correct.*

Subproblem 1.2 [1 pt] Assume that vector $\vec{v} \in \mathbb{R}^3$ is a scalar multiple of a vector $\vec{w}$, i.e. $\vec{v} = \lambda\vec{w}$ with some $\lambda \neq 0$. Prove that the length of vector $\vec{v}$ is λ times the length of vector $\vec{w}$.

Solution/comments. *We know that $\vec{v} = (v_1, v_2, v_3) = (\lambda w_1, \lambda w_2, \lambda w_3)$ and that the length of a vector is defined as the square root of the sum of the square of each of its coefficients, i.e. $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.*

To prove the statement, we just have to replace the v_i's with the λw_i's and do some simple arithmetic transformations:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(\lambda w_1)^2 + (\lambda w_2)^2 + (\lambda w_3)^2} = \sqrt{\lambda^2 w_1^2 + \lambda^2 w_2^2 + \lambda^2 w_3^2}$$
$$= \sqrt{\lambda^2 (w_1^2 + w_2^2 + w_3^2)} = \lambda\sqrt{w_1^2 + w_2^2 + w_3^2} = \lambda\|\vec{w}\|$$

Q.E.D

Subproblem 1.3 [1 pt] Assume that $\vec{v}$ and $\vec{w}$ are two unit vectors in $\mathbb{R}^3$.

2

(a) What do we know about $\vec{v}$ and $\vec{w}$ if their scalar product is zero, i.e. if $\vec{v} \cdot \vec{w} = 0$?
Shortly explain your answer.

(b) What do we know about $\vec{v}$ and $\vec{w}$ if their scalar product is one, i.e. if $\vec{v} \cdot \vec{w} = 1$?
Shortly explain your answer.

Solution/comments. *We know that the scalar product of two vectors* $\vec{v}$ *and* $\vec{w}$ *can be written as* $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$ *where* θ *is the angle between the two vectors. In this case, both vectors are unit vectors, i.e.* $\|\vec{v}\| = \|\vec{w}\| = 1$*, so the value of the scalar product simply becomes the cosine of* θ.

Because the cosine is zero for $\theta = 90°$ *and* $\theta = 270°$*, we know that in (a), the two vectors are perpendicular to each other.*

Because the cosine is one for $\theta = 0°$ *, we know that in (b), the two vectors are pointing in the same direction. Because they are both unit vectors, they are even identical.*

Problem 2: Basic geometric entities

Subproblem 2.1 [1 pt] Assume the following two points in $\mathbb{R}^3$:

$$\vec{p}_0 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \vec{p}_1 = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$$

Give a parametric representation of a line in $\mathbb{R}^3$ that goes through these two points.
(Note: write down what you are doing and shortly explain the single steps, so we see that you understand what you are doing and also that we can give you some credits even if your solution is wrong.)

Solution/comments. *The parametric representation of a line in 2D is* $\vec{p}(t) = \vec{p}_0 + t(\vec{p}_1 - \vec{p}_0)$ *where* $\vec{p}_0$ *and* $\vec{p}_1$ *are two points on the line. Note: these can be any points on the line. If we choose them like in the problem description, we get*

$$\vec{p}(t) = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Subproblem 2.2 [1.5 pt] Assume the following three points in $\mathbb{R}^3$:

$$\vec{q}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{q}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \vec{q}_2 = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

Give an implicit representation of a plane in $\mathbb{R}^3$ that goes through these three points.
(Note: write down what you are doing and shortly explain the single steps, so we see that you understand what you are doing and also that we can give you some credits even if your solution is wrong.)

Solution/comments. *The implicit representation of a plane (in vector format) in 3D is* $\vec{n}(\vec{p} - \vec{p}_0) = 0$ *where* $\vec{p}_0$ *is a random point on the plane,* $\vec{n}$ *is a normal vector to the plane, and* $\vec{p}$ *is the "implicit" parameter. We already have three points on the plane, so all we have to do is to calculate a normal vector and insert the values into the implicit plane equation.*

We can calculate the normal vector by taking the cross product of two vectors on the plane. These can be random vectors, so we take for example $\vec{q}_1 - \vec{q}_0$ *and* $\vec{q}_2 - \vec{q}_0$*. (Note: any other vector on the plane will do it. However,* not *the vectors* $\vec{q}_i$ *alone, because those are not vectors* on *the plane but vectors pointing to locations on the plane!).*

With this we get

$$\vec{v} = \vec{q}_1 - \vec{q}_0 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \textit{ and } \vec{w} = \vec{q}_2 - \vec{q}_0 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

The cross product of these two vectors is

$$\vec{v} \times \vec{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 - 1 \cdot 2 \\ 1 \cdot 2 - 1 \cdot 2 \\ 1 \cdot 2 - 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

This is our normal vector and if we chose for example $\vec{q}_0$ as vector $\vec{p}_0$ in the general equation above, we get

$$\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} (\vec{p} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) = 0$$

Of course, this solution is not unique, because $\vec{p}_0$ can be any vector pointing to a location on the plane. In addition, the normal vector is not unique (but has to be a scalar multiple of the solution presented here).

Subproblem 2.3 [1 pt] Calculate the intersection of the plane and the line that you created in the preceeding two subproblems. What is the geometric interpretation of your solution?

What other options do exist when you intersect a line with a plane in 3D (indicate the number of possible solutions and the geometric interpretation of it)?

Solution/comments. *Our plane equation is*

$$\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} (\vec{p} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) = 0$$

and our line is

$$\vec{p}(t) = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

The intersection are all points that fulfil both equations. When we put the line equation into the plane equation, we get

$$\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} (\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) = 0$$

Now, we have an equation with one parameter t. Solving for t leads to

$$\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4 + 4t = 0$$

what gives us $t = -1$. Putting this value into the line equation give us our solution, which is

$$\vec{p}(-1) = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

We see that this is a point in 3D, so the line and the plane intersect in one single point.

Other possible options when looking for the intersection of a line with a plane in 3D are

(i) the line is on the plane: then we get an infinite number of solutions (i.e. all points describing the line)

(ii) the line is parallel to the plane (but not on it): then we get no solution because the line and the plane do not intersect with each other

Subproblem 2.4 [1.5 pt] One characteristic of a parametric equation is that it is controlled by less parameters than dimensions in the space. For example, for the parametric equation of a sphere in 3D, we have two controlling parameters (or in other words: we are able to uniquely describe and address each 3-dimensional point on the sphere by specifying just two parameters).

(a) What is the controlling parameter in the line equation that you created in the first subproblem? Explain how it can be used to uniquely describe each point on the 2-dimensional line. (Hint: it might help to describe the geometric interpretation of the components of the whole parametric equation first).

(b) Write down the general form of a parametric equation for a circle around the origin in 2D. What is the controlling parameter in this equation? Shortly explain how it can be used to describe every point on the circle. (Hint: it might help to draw such a circle and a vector that points from the origin to a random point on the circle.)

Solution/comments. *(Note: The following is a very detailed description and obviously, various other options exist to give a correct answer here.)*

(a) Generally, a parametric quation of a line is characterized by two vectors and a controlling parameter, which is obviously t in the equation given above. The first vector points to a position on the line. It is also called support vector *because it somehow fixes or "supports" the position of the line in the 3D space. The other vector is on the line. It is also called* direction vector *because it specifys the direction or orientation of the line within the 3D space. Arithmetically, t is multiplied with the vector, so it changes it's length. If $t < 0$ it also makes the resulting vector point in the opposite direction. However, other than that it doesn't change the orientation of the vector in the 3D space. Since t can take any value, you can use it to create vectors of any length and thus describe any point on the line.*

Informally, you could also describe this as: "Go along the support vector to get to an initial position on the line. Then follow the direction vector for a distance proportional to t to get to any position on the line."

(b) The parametric equation of a circle in 2D is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

where r is the radius of the circle and obviously ϕ is the controlling parameter. If you draw such a circle and a vector pointing to a random point $\vec{p} = (x,y)$ on the circle, you see that the vector forms a triangle with the x-axis where the angle at the origin is ϕ, the other angle on the x-axis is $90°$, and the length of the three sides are the radius r (which is equal to the length of $\vec{p}$) and x and y (which are the coefficicients of $\vec{p}$). Note: this would have been a good way to come up with the circle equation above in the first place, in case you didn't remember it out of your head. From the image we can see that by modifying the angle ϕ, we can "turn" $\vec{p}$ around the whole circle. Because the length of $\vec{p}$ stays the same and is equal to the radius of the circle, we can therefore describe each random point on the circle by a unique angle ϕ.

Again, this is a very detailed description and much less was needed to get the maximum credits for this subproblem.

Problem 3: Matrices

Subproblem 3.1 [1 pt] Assume that A and B are two $n \times n$ matrices. (Note: this also means that matrix multiplication between them is defined.)

Are the following statements correct or not? Give a proof of your answer in both cases.

1. $AB = BA$
2. $A(BC) = A(CB)$

Solution/comments. *The first statement is wrong. We can prove this by giving a simple counterexample where at least one of the coefficients from the resulting matrix on the left side differs from the corresponding one on the right side.*

The second statement is also wrong. Maybe the easiest way to prove this is to choose the identity matrix for A. Then the equation becomes equivalent with the first statement and we can use the same matrices as above as a counterexample.

Subproblem 3.2 [1.5 pt] Calculate the inverse A^{-1} of the following matrix using Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Note: you have to use Gaussian elimination to solve this problem (or the "forward-backward-step" variation that was mentioned in the lecture), otherwise you will get no credits (even if your result is correct).
Write down each step, so we can understand what you were doing and also can give you some credits even if your solution is wrong.

Solution/comments.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -4 & -3 & 1 & 1 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & -1/4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & -1/4 & 3/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & -1/4 \end{array}\right)$$

Hence, the result is the following matrix:

$$A^{-1} = \begin{pmatrix} -1/4 & -1/4 & 3/4 \\ -1/4 & 3/4 & -1/4 \\ 3/4 & -1/4 & -1/4 \end{pmatrix} = 1/4 \begin{pmatrix} -1 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & -1 \end{pmatrix}$$

Subproblem 3.3 [1.5 pt] Assume we have three linear equations - each with three unknown parameters. We can interpret each of these equations as a plane in 3D. Solving the resulting system of linear equations (e.g.) by Gaussian elimination gives us the intersection of these three planes.

What kind of solutions can we get? For each of the possible cases: describe what happens when we solve the linear equation system with Gaussian elimination and discuss *all* possible geometric interpretations.

Hints: if you have problems thinking of all possible geometric interpretations, it might help to first think about what you can get when you intersect two planes, and then what possibilities exist if you calculate the intersection of that result with the remaining third plane.

Solution/comments. *Considering Gaussian elimination, there are three possible cases:*

First, we can get one single solution, i.e. a point in 3D. Geometrically, this means that the first two planes intersect in a line and that this line "goes through" the third plane (i.e. is not parallel to it), thus intersecting in a single point.

Second, we can get an equation such as $0x + 0y + 0z = 0$ *which is true for all values of x, y, and z. Hence, we have three unknown parameters and only two equations. We are not able to solve this, but we can use one equation to eliminate the third parameter and the second equation to express one parameter in relation to the other one. Geometrically, this leads to the equation of a line. This means, all planes intersect in a single line.*
In case we have two equations like $0x + 0y + 0z = 0$, *we only have one equation with three parameters. This is a single plane and means that all three planes are identical.*

Third, we can get an equation such as $0x + 0y + 0z = b$ *with* $b \neq 0$. *This can not be solved for any value of x, y, and z and hence, we get no solution. Geometrically, this means that the intersection set is empty.*
There are different options how this can happen. First, two planes can be parallel to each other. (Note that it doesn't matter how the third one intersects with them. There are three options, but all of them have an empty intersection. In case you dare, these three options are: 3rd is identical to one of the others, 3rd is parallel to both of them, 3rd intersects with both in a line but those two intersection lines are parallel)
Second, all three planes pairwise intersect in one line, but these intersection lines are in turn parallel to each other.

Subproblem 3.4 [1 pt] One way to calculate determinants is via cofactors. Calculate the cofactor a^c_{12} for the coefficient a_{12} of the following matrix:

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

Note: you do *not* have to calculate the whole determinant, but just the one cofactor for a_{12}. Write down each step, so we can understand what you were doing and also can give you some credits even if your solution is wrong.

Solution/comments. *To calculate the cofactor of a coefficient* c_{ij}, *we cross out the i-th row and j-th column of the matrix, calculate the determinant of the resulting matrix and multiply it with* $(-1)^{i+j}$. *Since the resulting matrix here is a* 2×2 *matrix, calculating its determinant is easy (either by recursion or by just writing it down if you remember the equation for it).*

For a_{12} we therefore get

$$a^c_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} = -1(24-30) = 6$$

Problem 4: Transformations

Subproblem 4.1 [1 pt] A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a linear transformation if it satisfies certain conditions. What are these conditions?

Solution/comments. *There are two conditions that a transformation has to satisfy in order to be a linear transformation:*

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ *for all* $\vec{u}, \vec{v} \in \mathbb{R}^n$.
2. $T(c\vec{v}) = cT(\vec{v})$ *for all* $\vec{v} \in \mathbb{R}^n$ *and all scalars* c.

These can alternatively be summarized in a single condition, which is:

1. $T(c_1\vec{u}+c_2\vec{v}) = c_1T(\vec{u}) + c_2T(\vec{v})$ *for all* $\vec{u}, \vec{v} \in \mathbb{R}^n$ *and all scalars* c_1, c_2.

Subproblem 4.2 [1 pt] Prove that scaling in 2D, i.e. multiplication of a vector $\vec{v} = (x, y)$ with the following matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

is a linear transformation. (Note: to do this, you should use what you wrote down in the preceeding subproblem)

Solution/comments. *To prove that scaling in 2D is a linear transformation is straightforward. We just have to write the general case down and verify that it indeed satisfies the conditions specified in the preceeding subproblem (either the first two or the third one).*

If we chose the third one, we have for every random vector $c_1\vec{u}+c_2\vec{v}$ *(left side of third equation above):*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c_1u_x + c_2v_x \\ c_1u_y + c_2v_y \end{pmatrix} = \begin{pmatrix} a(c_1u_x + c_2v_x) + 0 \\ 0 + b(c_1u_y + c_2v_y) \end{pmatrix} = \begin{pmatrix} ac_1u_x + ac_2v_x \\ bc_1u_y + bc_2v_y \end{pmatrix}$$

which is indeed the same as (right side of equation above):

$$c_1 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} + c_2 \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = c_1 \begin{pmatrix} au_x \\ bu_y \end{pmatrix} + c_2 \begin{pmatrix} av_x \\ bv_y \end{pmatrix} = \begin{pmatrix} ac_1u_x + ac_2v_x \\ bc_1u_y + bc_2v_y \end{pmatrix}$$

(Note: alternatively, you could have proven the first two conditions)

Subproblem 4.3 [2 pt] In $\mathbb{R}^2$, the matrix

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

defines a counterclockwise rotation by the angle ϕ around the origin.

(a) What is the rotation matrix for *clockwise* rotation around the origin in $\mathbb{R}^2$?
Shortly explain how you got your solution.

(b) Give a 3x3 transformation matrix for a similar rotation around the z-axis in $\mathbb{R}^3$.
Shortly explain how you got your solution.

Solution/comments. *For (i), there are two obvious ways to get the solution. The first one is to remember that the columns of the transformation matrix are the images of the base vectors. By drawing an image and looking at the trigonometric relationships we can easily see the solution. Alternatively, we can argue that clockwise rotation is the same as counterclockwise rotation if we just take the negative angle, i.e. replace* ϕ *in the matrix above with* $-\phi$*. Because* cos *is symmetric to the y-axis (i.e.* $\cos\phi = \cos -\phi$*) and* sin *is point symmetric to the origin (i.e.* $\sin\phi = -\sin -\phi$*), we get the following matrix:*

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

For (ii), we can argue that the z-values don't change when we do such a rotation and that the x- and y-values behave in a similar way as if we do rotation in the $x-y$*-plane. Hence, we just have to add a third row and column to the 2D rotation matrix given in the problem description that doesn't change the z-values (but does the same with the x- and y-values). This leads to the following matrix:*

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subproblem 4.4 [1 pt] Describe in your own words what happens to a point $\vec{p}$ in $\mathbb{R}^3$ if you apply the following transformation matrix to it:

$$\begin{pmatrix} -1 & 0 & 0 & x_m \\ 0 & 1 & 0 & y_m \\ 0 & 0 & 1 & z_m \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

How are the values in the last row of this matrix called and why do we need them?

Solution/comments. *If we write this down for a random point in* $\mathbb{R}^3$*, i.e.*

$$\begin{pmatrix} -1 & 0 & 0 & x_m \\ 0 & 1 & 0 & y_m \\ 0 & 0 & 1 & z_m \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -x + x_m \\ y + y_m \\ z + z_m \\ 1 \end{pmatrix}$$

then we see that a constant value gets added to the first three coefficients of the vector, what means that it gets translated to a new position. We also see, that the the x-coordinate changes its sign, whereas the y- and z-values are not changed (besides the constant factor that is added, of course). This transformation is a reflection on the $y-z$*-plane in* $\mathbb{R}^3$ *(realized by the* 3×3 *submatrix in the upper left) combined with a translation (realized by the values in the right column).*

For the latter one, we need the values in the last row, because translation is not a linear transformation and thus can not be done with "normal" matrix multiplication. However, by adding this so called homogeneous coordinates*, we are able to add constant factors to the first three coefficients (because by going one dimension higher, the operation becomes similar to shearing which is again a linear transformation).*