Graphics 2007/2008

Midterm Exam

Thu, Dec 13, 2007, 13:15–15:00

Solutions (sketch) and comments

Errors and omissions excepted!

Problem 1: Vectors, basic geometric shapes, intersections

Subproblem 1.1 [2 pt] Which of the following answers are correct? Write down *all* possible solutions (i.e. there might be more than one correct answer to each question!). It is *not* needed to give an explanation.

The intersection of ...

(i)	... a line and a sphere can have exactly	[a] 0	[b] 1	[c] 2	or	[d] ∞	solutions
(ii)	... a line and a plane can have exactly	[a] 0	[b] 1	[c] 2	or	[d] ∞	solutions
(iii)	... a plane and a plane can have exactly	[a] 0	[b] 1	[c] 2	or	[d] ∞	solutions
(iv)	... three planes can have exactly	[a] 0	[b] 1	[c] 2	or	[d] ∞	solutions

(i) 0 if the line misses the sphere
1 if the line "touches" the sphere but doesn't cross it
2 if line and sphere intersect

(ii) 0 if they are parallel
1 if they intersect
∞ if the line lies on the plane

(iii) 0 if they are parallel
∞ if they are identical or intersect in a line

(iv) If the intersection of the first 2 planes (cf. iii) ...
... has 0 solutions: 0 solutions for intersect. w. 3rd plane as
... intersect in a line: 0, 1, or ∞ solutions (cf. ii)
... is a plane: 0 or ∞ solutions (cf. iii)

Note: It was <u>not</u> necessary to give an explanation here!

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Subproblem 1.2 [4 pt] Assume the following three points in $\mathbb{R}^3$:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

(1.2a) Give a parametric equation of the line l through the points **a** and **b**.

(1.2b) What is the geometric interpretation of the parametric equation given in (1.2a)?

(1.2c) Calculate a normal vector **n** for the plane defined by **a**, **b**, and **c**

(1.2d) Create an implicit representation of the plane defined by **a**, **b**, and **c**.
(Note: "Create" means that it is not sufficient to just write down the equation of the plane but that we should be able to recognize how you got this solution.)

a) $\vec{p}(t) = \vec{a} + t(\vec{b} - \vec{a}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ (Note: Other solutions exist

b) There are several ways to give a well-defined specification of a line, eg. via two points on the line, a normal vector and a point on the line, or a direction and a point on the line.

The parametric equation is related to the latter case. Hence, any answer including the keywords "point/position" and "direction/orientation" usually got full marks.

Note: Some people misinterpreted this question and tried to describe the appearance of this particular line in $\mathbb{R}^3$. If the decription was correct and reasonable, they also got full marks

c) The cross-product of any 2 vectors on the plane will give a correct solution, eg.

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \\ 4 \end{pmatrix}$$

d) Impl. repres. of a plane: $ax + by + cz - d = 0$
with $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ being a normal vector of the plane, eg. the one calculated in c).

To get the correct value for d, we can just insert any point on the plane, eg. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ which gives us

$$-6 - 6 + 4 - d = 0 \Rightarrow d = -8 \Rightarrow \boxed{-6x - 6y + 4z + 8 = 0}$$

Problem 2: Matrices

Subproblem 2.1 [1 pt] Prove that matrix multiplication is not commutative, i.e. that in general $\mathbf{AB} \neq \mathbf{BA}$.

Maybe the easiest way to show this, is to use an example where the dimensions don't match, eg.

$$\begin{array}{cc} & \begin{pmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{pmatrix} \\ (1\ 2) & \cdots \end{array} \checkmark \quad \text{but} \quad \begin{array}{cc} & (1\ 2) \\ \begin{pmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{pmatrix} & \end{array}$$

is undefined

In the lecture, I used $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}$ and showed that a_{11} is different for both ~~cases~~ cases.

Subproblem 2.2 [3 pt] Assume the following three planes in $\mathbb{R}^3$:

$$\begin{aligned} x+2y+8z &= 11 \\ x+4y+12z &= 17 \\ 4y+10z &= 14 \end{aligned}$$

(2.2a) Construct all intersection points of the three planes using Gaussian elimination.

(2.2b) What is the geometric interpretation of your solution?

a) ... (straight forward) The correct solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

b) All 3 planes intersect in a single point, ie. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(Note: If you got a wrong result in a) but gave the correct interpretation of it in b), you got full mar

Subproblem 2.3 [2 pt] Show that for a nxn-matrix **A**, $\mathbf{AA}^T$ is a symmetric matrix (i.e. you have to show that $c_{ij} = c_{ji}$ for any coefficient c_{ij} of the matrix $\mathbf{AA}^T$).

There are different ways to show this, eg.:

Let a_{ij} be the coefficients of matrix A and a^*_{ij} the ones of matrix A^T.

Then: $a_{ij} = a^*_{ji}$ (equ. A)

(and vice versa, obviously)

Hence:

$$
\begin{aligned}
c_{ij} &= a_{i0} \cdot a^*_{0j} + \; \dots \; + a_{in} \cdot a^*_{nj} && \text{equ. A: } a^*_{ij} = a_{ji} \\
&= a_{i0} \cdot a_{j0} + \; \dots \; + a_{in} \cdot a_{jn} && \text{equ. A: } a_{ij} = a^*_{ji} \\
&= a^*_{0i} \cdot a_{j0} + \; \dots \; + a^*_{ni} \cdot a_{jn} \\
&= a_{j0} \cdot a^*_{0i} + \; \dots \; + a_{jn} \cdot a^*_{ni} \\
&= c_{ji} \quad \text{f.a. } i,j \quad \checkmark
\end{aligned}
$$

<u>Note</u>: Many people just demonstrated one particular case (or even a particular example) and then said, that it is obvious or trivial that this case generalizes to random values of n.

Although giving a single example (or some examples) is obviously <u>not</u> enough to prove a general case, we were generous and still gave 0.5 to 1.5 marks, depending on the quality of your particular representation.

Problem 3: Transformations

Subproblem 3.1 [2 pt] Assume the following transformation matrices:

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Associate each of these matrices with one of the following statements: (Note: You only have to give the letter of the correct matrix **A**, **B**, **C**, or **D** for each statement, i.e. an explanaition is *not* needed.)

This transformation matrix represents ...

(i) ... a reflection on $y = 0$

(ii) ... a reflection on $x = 0$

(iii) ... a reflection on $x = y$

(iv) ... a point reflection in the origin

Solution:

i → B

ii → A

iii → D

iv → C

Note: There was an unexpected high number of mistakes for (i) and (ii). Unfortunately, many people did not realize that $y=0$ represents the x-axis and $x=0$ the y-axis and therefore confused the two results.

Subproblem 3.2 [1 pt] Write down the 3x3 matrix for a rotation by an angle of θ around the x-axis in $\mathbb{R}^3$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

<u>Note</u>: Misplacing the minus sign in front of the sins is also correct since it just represents a rotation in the other direction (but nevertheless still around the x-axis)

Subproblem 3.3 [1 pt] Describe in your own words what happens to a vector **v** if you apply the following transformation matrix to it:

$$\begin{pmatrix} 2 & 0 & 0 & x_m \\ 0 & 2 & 0 & y_m \\ 0 & 0 & 2 & z_m \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

The matrix is a combination of a scaling matrix and a trans~~formation~~lation matrix (x_m, y_m, z_m and homogeneous coords.).

Since a vector has no location, ~~it~~ can not be "moved", so the correct answer would have been sth. like "... gets scaled by factor 2".

Unfortunately, we have been a little imprecise in the formulation of this question, ie. did not ~~we~~ mention that it is a vector in $\mathbb{R}^3$, which is why we were generous in the grading and gave everyone answering sth. like "... gets scaled by factor 2 and translated by x_m, y_m, and z_m in x-, y-, and z-direction, respectively" full marks as well.

Subproblem 3.4 [2 pt] The following matrix defines scaling (in $\mathbb{R}^2$) by a factor of a and b in x- and y-direction, respectively:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Prove that matrix multiplication with this scaling matrix is a linear transformation.

Unfortunately, not many people came up with correct solutions to this problem, although it was quite easy.

Just write down the definition of a linear transformation, i.e.

$$\text{(i)} \quad T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$\text{(ii)} \quad T(c \cdot \vec{v}) = c \cdot T(\vec{v})$$

and then show this for the concrete example.

E.g. in the second case you get:

$$T(c \cdot \vec{v}) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c v_1 \\ c v_2 \end{pmatrix} = \begin{pmatrix} a c v_1 \\ b c v_2 \end{pmatrix} = c \begin{pmatrix} a v_1 \\ b v_2 \end{pmatrix}$$

$$= c \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = c \cdot T(\vec{v}) \quad ✓$$

(and the same for (i))