Graphics 2010/2011

T1

Midterm exam

Tue, May 24, 2011

Do not open this exam until instructed to do so.

Read the instructions on this page carefully.

- You may write your answers in English, Dutch, or German.
  Use a pen, not a pencil. Avoid usage of the color red.
- Write down your name and student number on every paper you want to turn in.
  Additional paper is provided by us. You are not allowed to use your own paper.
- You may **not** use books, notes, or any electronic equipment
  (including your cellphone, even if you just want to use it as a clock).

- You have **max. 2 hours** to work on the questions.
  If you finish early, you may hand in your work and leave,
  except for the first half hour of the exam.
- When you hand in your work, have your student ID ready for inspection
  and write your name and student number on the list of participants.

- The exam consists of 4 problems printed on 5 pages (including this one).
  It is your responsibility to check if you have a complete printout.
  If you have the impression that anything is missing, let us know.

Good luck!

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Problem 1: Vectors

■ **Subproblem 1.1 [2 pts]:** Assume two vectors $\vec{a} = (1,0)$ and $\vec{b} = (0,1)$.
Which of the following statements are correct?

Note: This is a multiple choice question. No explanation is required. Just list all correct answers.

1. $\vec{a}$ and $\vec{b}$ have the same length.
2. $\vec{a}$ and $\vec{b}$ have the same direction.
3. $\vec{a}$ and $\vec{b}$ are linearly independent.
4. $\vec{a}$ and $\vec{b}$ form a 2D basis.
5. The inner product of $\vec{a}$ and $\vec{b}$ is 1.
6. $\vec{a}$ is a scalar multiple of $\vec{b}$.
7. $\vec{a}$ is a normal vector to $\vec{b}$.
8. $\vec{a}$ and $\vec{b}$ form an orthonormal basis.

■ **Subproblem 1.2 [2 pts]:** Assume three vectors $\vec{c} = (1,2)$, $\vec{d} = (-2,1)$, and $\vec{e} = (3,6)$.
Which of the following statements are true?

Note: This is a multiple choice question. No explanation is required. Just list all correct answers.

1. $\vec{c}$ and $\vec{d}$ have the same length.
2. $\vec{c}$ and $\vec{e}$ have the same direction.
3. The inner product of $\vec{c}$ and $\vec{d}$ is 1.
4. $\vec{c}$ and $\vec{d}$ form a 2D basis.
5. $\vec{c}$ and $\vec{e}$ form an orthnormal basis.
6. $\vec{c}$ and $\vec{e}$ are parallel.
7. $\vec{c}$ is the transposed of $\vec{d}$.
8. $\vec{c}$ is a scalar multiple of $\vec{e}$.

■ **Subproblem 1.3 [2 pts]:** Assume three vectors $\vec{a} = (x_a, y_a)$, $\vec{b} = (x_b, y_b)$, and $\vec{c} = (x_c, y_c)$ in $\mathbb{R}^2$.
Prove that the inner product (also known as dot product or scalar product) of these three vectors is distributive over addition, i.e. show that

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

■ **Subproblem 1.4 [1 pts]:** Let $\vec{v} = (x,y)$ be a random vector in $\mathbb{R}^2$.
Prove that $\vec{v}^* = (-y,x)$ is orthogonal to $\vec{v}$.

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Problem 2: Basic geometric entities

Assume the following three vectors in $\mathbb{R}^3$ that we will use in the following subproblems:

$$\vec{p}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{p}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \vec{p}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

■ **Subproblem 2.1 [1 pts]:** Give a parametric equation of the line that goes through the points represented by the vectors $\vec{p}_0$ and $\vec{p}_1$.

■ **Subproblem 2.2 [1.5 pts]:** Give an implicit equation of a plane that goes through the points represented by the vectors $\vec{p}_0$, $\vec{p}_1$, and $\vec{p}_2$.

■ **Subproblem 2.3 [1.5 pts]:** Give an implicit equation for a sphere that has its center around the point represented by $\vec{p}_0$ and where $\vec{p}_1$ is on the sphere's surface.

■ **Subproblem 2.4 [2 pts]:** Calculate the intersection of the sphere you created in the last subproblem and the line $\ell = (0,1,0) + s(0,0,\sqrt{2})$. What is the geometric interpretation of your solution? What other possible solutions can you get when you calculate the intersection between a line and a sphere? Explain why you can get these solutions and the geometric interpretation of it.

■ **Subproblem 2.5 [1 pts]:** The implicit equation of a plane that you created in subproblem 2.2 splits $\mathbb{R}^3$ into a positive and a negative half-space. Why are these two half-spaces called *positive* and *negative*? In which half-space are the following points: $\vec{a} = (2,2,2)$, $\vec{b} = (1,1,1)$, and $\vec{c} = (0,0,0)$? Explain your answer. *Note: A short informal explanation is sufficient to get full credits. A full formal prove is not needed for this subproblem.*

Problem 3: Matrices

■ **Subproblem 3.1 [2 pts]:** Assume the following three matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

Answer the following questions:

1. Which of the three matrices is a diagonal matrix (if any)?
   Shortly explain your answer.
2. A is the inverse matrix of C. Is this correct?
   Shortly explain your answer.
3. C is the inverse matrix of A. Is this correct?
   Shortly explain your answer.
4. Use two of the three matrices to prove that matrix multiplication is not commutative.

■ **Subproblem 3.2 [1 pts]:** Prove that $(sA)^T = sA^T$ for all scalar values s and any matrix A.

■ **Subproblem 3.3 [2 pts]:** Assume the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

1. Calculate the determinant of A.
2. Calculate the cofactor of a_{23}.

■ **Subproblem 3.4 [2 pts]:** Use Gaussian elimination to calculate the inverse A^{-1} of the following matrix:

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 4 & -2 & 4 \\ 2 & 4 & 2 \end{pmatrix}.$$

Note: Write down each step, so we can give you at least some credit even if your result is wrong due to some calculation errors.

■ **Subproblem 3.5 [1 pts]:** The intersection of three planes in $\mathbb{R}^3$ can either be a single point, a line, or it is empty if the planes don't intersect. If we use Gaussian elimination to calculate the possible intersection, what happens in the latter case, i.e. an empty set of intersections? What is the geometric interpretation of this, i.e. how can the planes look like if their intersection set is empty?

Problem 4: Transformations

■ **Subproblem 4.1 [1.5 pts]:** T is a transformation matrix in 2D:

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

1. What kind of transformation does it realize?
2. Prove that it is a linear transformation.

■ **Subproblem 4.2 [2 pts]:** In this subproblem, we look at transformations in 2D.

1. Give a matrix for a counterclockwise rotation around the origin about a random angle ϕ.
2. Give a matrix for a translation from point $(2,2)$ to the origin.
3. Give a matrix for a counterclockwise rotation around point $(2,2)$ about a random angle ϕ.
   Shortly explain how you got your answer.

■ **Subproblem 4.3 [2 pts]:** Look at the following transformation matrix T:

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. If we apply this transformation matrix to an object in 3D, what happens to the object?
2. Modify this matrix, so it does not only apply this operation but also doubles the size of the object with respect to the origin.

■ **Subproblem 4.4 [1.5 pts]:** Look a the following transformation matrix T:

$$T = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

1. And what happens if we apply it to a vector $\vec{v} = (x, y, z)$ in **3D**?
2. What happens if we apply this transformation matrix T to a location (x, y) in **2D**
3. In the previous case, we used a 3×3 matrix to do a transformation in 2D. Explain why.

   Note: In order to get full credits, it is sufficient to say how the coefficients in the last row are called and why we need them.