Graphics 2010/2011

T1

Midterm exam

Tue, May 24, 2011

COMMENTS AND SOLUTIONS

No responsibility is taken for the correctness of the provided information.

Note that there can be more than one correct solution for most problems. Also, the following information should not be considered standard solution, but rather as comments that generally go way beyond what was required to get the maximum credit for a particular subproblem.

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Problem 1: Vectors

■ **Subproblem 1.1 [2 pts]:** Assume two vectors $\vec{a} = (1,0)$ and $\vec{b} = (0,1)$.
Which of the following statements are correct?

Note: This is a multiple choice question. No explanation is required. Just list all correct answers.

1. $\vec{a}$ and $\vec{b}$ have the same length.
2. $\vec{a}$ and $\vec{b}$ have the same direction.
3. $\vec{a}$ and $\vec{b}$ are linearly independent.
4. $\vec{a}$ and $\vec{b}$ form a 2D basis.
5. The inner product of $\vec{a}$ and $\vec{b}$ is 1.
6. $\vec{a}$ is a scalar multiple of $\vec{b}$.
7. $\vec{a}$ is a normal vector to $\vec{b}$.
8. $\vec{a}$ and $\vec{b}$ form an orthonormal basis.

■ **Solution/comments:** If you remembered that these vectors are the basis of the carthesian coordinates system in 2D, you should immediately see that 1., 3., 4., 7., and 8. are true and 2., 5., and 6. are wrong. If not, you have to think a little more:

1. The length of a vector is defined by $\|\vec{v}\| = \sqrt{x^2 + y^2}$. Hence, both vectors have the length 1, so this statement is **correct**.
2. This statement is **wrong** and there are multiple ways to see this. For example, they only point in the same direction if they are parallel (which they are not), if they are a scalar multiple of each other (which is the same as being parallel), if their inner product is 1 (which is is not), etc.
3. This statement is **correct** and again, there are multiple ways to see this: They are not parallel (i.e. there is no λ such that $\vec{a} = \lambda\vec{b}$), they don't point in the same direction (cf. 2.), etc.
4. Because 3. is correct, this statement must be **correct**, too.
5. The inner product of two vectors is defined by $\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$ which for our values of $\vec{a}$ and $\vec{b}$ is 0 and not 1, so this statement is **wrong**.
6. In 2. and 3., we already saw that they are not parallel, so this is also **wrong**.
7. This is **correct**. The vectors are perpendicular to each other because their inner product is 0 (cf. above).
8. To form an orthonormal basis, they must be unit vectors that are perpendicular to each other. From 1. and 5. we know that this is indeed the case here, so this statement is **correct**.

■ **Subproblem 1.2 [2 pts]:** Assume three vectors $\vec{c} = (1,2)$, $\vec{d} = (-2,1)$, and $\vec{e} = (3,6)$.
Which of the following statements are true?

Note: This is a multiple choice question. No explanation is required. Just list all correct answers.

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1. $\vec{c}$ and $\vec{d}$ have the same length.
2. $\vec{c}$ and $\vec{e}$ have the same direction.
3. The inner product of $\vec{c}$ and $\vec{d}$ is 1.
4. $\vec{c}$ and $\vec{d}$ form a 2D basis.
5. $\vec{c}$ and $\vec{e}$ form an orthnormal basis.
6. $\vec{c}$ and $\vec{e}$ are parallel.
7. $\vec{c}$ is the transposed of $\vec{d}$.
8. $\vec{c}$ is a scalar multiple of $\vec{e}$.

■ **Solution/comments:** It should be easy to see that $\vec{c}$ and $\vec{e}$ are scalar multiples of each other (with λ being either 3 or $1/3$). Also, $\vec{c}$ and $\vec{d}$ are clearly linearly independent (and so are $\vec{e}$ and $\vec{d}$ of course). $\vec{c}$ and $\vec{d}$ are also perpendicular (multiple ways to see this: e.g by calculating the inner product, by remembering that $(-a, 1)$ is a normal vector for $(1, a)$, by drawing it, etc.).

1. The length of a vector is defined by $\|\vec{v}\| = \sqrt{x^2 + y^2}$. Because of the square root, this obviously leads to the same result for both vectors, so this statement is **correct**.
2. Because they are scalar multiples of each other, this must be **correct**.
3. No, it is 0 (because they are perpendicular to each other), so this is **wrong**.
4. This is obviously **correct** because they are linarly independent.
5. They don't even form a basis because they are parallel, so this is clearly **wrong**.
6. Yes, this is **correct** (see above).
7. If you look at vector notation, you see that this is obviously **wrong**.
8. Yes, **correct**, see above.

■ **Subproblem 1.3 [2 pts]:** Assume three vectors $\vec{a} = (x_a, y_a)$, $\vec{b} = (x_b, y_b)$, and $\vec{c} = (x_c, y_c)$ in $\mathbb{R}^2$. Prove that the inner product (also known as dot product or scalar product) of these three vectors is distributive over addition, i.e. show that

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

■ **Solution/comments:** To prove that this equation is always true, we just have to show that the left side is the same as the right side for three random vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

On the left side we have:
$(x_a, y_a) \cdot ((x_b, y_b) + (x_c, y_c))$
$= (x_a, y_a) \cdot (x_b + x_c, y_b + y_c)$
$= x_a(x_b + x_c) + y_a(y_b + y_c)$
$= x_a x_b + x_a x_c + y_a y_b + y_a y_c$

And on the right side we have:
$(x_a, y_a) \cdot (x_b, y_b) + (x_a, y_a) \cdot (x_c, y_c)$
$= x_a x_b + y_a y_b + x_a x_c + y_a y_c$

This is obviously the same.

Notice that the above statement is only true if (x,y) is not the nullvector. I forgot to mention this in the exercise, but fortunately it didn't have a negative impact on your gradings :)

■ **Subproblem 1.4 [1 pts]:** Let $\vec{v} = (x,y)$ be a random vector in $\mathbb{R}^2$.
Prove that $\vec{v}^* = (-y,x)$ is orthogonal to $\vec{v}$.

■ **Solution/comments:** There are multiple ways to prove this. Because we know that the inner product of two vectors is 0 if and only if they are orthogonal to each other, we can for example just calculate it:

$$(x,y) \cdot (-y,x) = x(-y) + yx = -xy + xy = 0$$

So, we see that the scalar product is 0. That proves that these are indeed two orthogonal vectors.

Problem 2: Basic geometric entities

Assume the following three vectors in $\mathbb{R}^3$ that we will use in the following subproblems:

$$\vec{p}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{p}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \vec{p}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

■ **Subproblem 2.1 [1 pts]:** Give a parametric equation of the line that goes through the points represented by the vectors $\vec{p}_0$ and $\vec{p}_1$.

■ **Solution/comments:** The general form of a parametric equation of a line through two points $\vec{p}_0$ and $\vec{p}_1$ in $\mathbb{R}^3$ is

$$\vec{p}(t) = \vec{p}_0 + t(\vec{p}_1 - \vec{p}_0)$$

Filling in the concrete values for $\vec{p}_0$ and $\vec{p}_1$ gives us

$$\vec{p}(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Notice that any scalar multiple of the vector $(2,0,0)$ for the direction vector would be correct, too, and that any other vector on the line could have been used as support vector instead of p_0.

■ **Subproblem 2.2 [1.5 pts]:** Give an implicit equation of a plane that goes through the points represented by the vectors $\vec{p}_0$, $\vec{p}_1$, and $\vec{p}_2$.

■ **Solution/comments:** In its general form, the implicit representation of a plane in $\mathbb{R}^3$ can either be written as

$$\vec{n}(\vec{p} - \vec{p}_0) = 0 \qquad \text{or as} \qquad ax + by + cz + d = 0$$

with $\vec{n}$ and (a,b,c) being normal vectors to the plane, and $\vec{p}_0$ being a point on the plane. So, in both cases, we need a normal vector. We can get that by taking the cross product of two vectors on the plane, e.g. $\vec{v} = \vec{p}_1 - \vec{p}_0 = (2,0,0)$ and $\vec{w} = \vec{p}_2 - \vec{p}_0 = (0,0,2)$ (notice: any other vector on the plane will do it, too; not however a vector representing a point on the plane!). The cross product of two vectors $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \times \vec{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 0 - 4 \\ 0 - 0 \end{pmatrix}$$

So, $\vec{n} = (0,-4,0)$ is a normal vector to the plane (and so is any scalar multiple of it). If we chose $\vec{p}_0$ as our point on the plane, we get for the first representation

$$\begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} (\vec{p} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}) = 0$$

If we decide to chose the second possible representation, we have $a = 0$, $b = -4$, and $c = 0$. To calculate d, we can just put any random point on the plane into this equation and solve for d. For example, we can take $\vec{p}_0$, then we get $0 - 4 * 1 - 0 + d = 0$, so our d is 4, and the solution is

$$-4y + 4 = 0$$

Notice that we can easily simplify this to $y = 1$, so this is a plane that is parallel to the Z-axis with a distance of 1 from it.

■ **Subproblem 2.3 [1.5 pts]:** Give an implicit equation for a sphere that has its center around the point represented by $\vec{p}_0$ and where $\vec{p}_1$ is on the sphere's surface.

■ **Solution/comments:** The general form of an implicit equation of a sphere in $\mathbb{R}^3$ is

$$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - r^2 = 0$$

The center (c_x, c_y, c_z) of our sphere is our vector $\vec{p}_0$, so we only need to calculate the value of its radius r. If $\vec{p}_1$ is on the sphere, than r is just the length of the vector $\vec{p}_1 - \vec{p}_0$, i.e. $\|(2,0,0)\| = \sqrt{2^2 + 0 + 0} = 2$. This gives us our solution

$$x^2 + (y - 1)^2 + z^2 - 4 = 0$$

Notice that you also got full credits if you used $\|\vec{p}_1 - \vec{p}_0\| - r = 0$ as implicit equation.

■ **Subproblem 2.4 [2 pts]:** Calculate the intersection of the sphere you created in the last subproblem and the line $\ell = (0,1,0) + s(0,0,\sqrt{2})$. What is the geometric interpretation of your solution? What other possible solutions can you get when you calculate the intersection between a line and a sphere? Explain why you can get these solutions and the geometric interpretation of it.

■ **Solution/comments:** Obviously, we can represent the line ℓ by the vector $(0,1,s\sqrt{2})$. Putting that into our sphere equation $x^2 + (y - 1)^2 + z^2 - 4 = 0$ gives us $0 + 0 + (s\sqrt{2})^2 - 4 = 0$, so $s^2 = 2$. That in turn gives us two possible values for s, namely $\sqrt{2}$ and $-\sqrt{2}$. Putting those into our line equation ℓ give us the two solutions $(0,1,2)$ and $(0,1,-2)$.

This means that the line goes through the sphere and intersects with it in two points (the ones we just calculated).

In general, when we calculate the intersection of a sphere and a line, we have to solve a quadratic equation – which can either have 0, 1, or 2 solutions. Geometrically, this can be interpreted as

- "the line and the sphere do not intersect" (no solution),
- "the line 'touches' the sphere" (or is a tangent to the sphere; 1 solution),
- "the line interesects with the sphere in two points" (as in our example above; 2 solutions).

■ **Subproblem 2.5 [1 pts]:** The implicit equation of a plane that you created in subproblem 2.2 splits $\mathbb{R}^3$ into a positive and a negative half-space. Why are these two half-spaces called *positive* and *negative*? In which half-space are the following points: $\vec{a} = (2,2,2)$, $\vec{b} = (1,1,1)$, and $\vec{c} = (0,0,0)$? Explain your answer.

Note: A short informal explanation is sufficient to get full credits. A full formal prove is not needed for this subproblem.

■ **Solution/comments:** If we put a random point $\vec{p}$ into our implicit equation (let's call it $f(x,y,z)$), the result will either be 0 (if $\vec{p}$ is on the plane), positive, or negative. For 2D and implicit lines, we have proven in the tutorials that all points with a positive result are on one side, and all with a negative result are on the other side. Hence, we call these two sides *positive* and *negative* half-space (the generalization to 3D is straightforward).

To solve the second part of the problem, just put those points into the equation and see if $f(x,y,z)$ is 0, positive, or negative. With $f(x,y,z) = -4y+4$ we get that

- $f(2,2,2) < 0$, so $(2,2,2)$ is in the negative half-space,
- $f(0,0,0) > 0$, so $(0,0,0)$ is in the positive half-space, and
- $f(1,1,1) = 0$, so $(1,1,1)$ is neither in the positive nor the negative half-space. It is on the plane.

Notice that if you have chosen a normal vector that points in the oposite direction (e.g. $(0,4,0)$ or any scalar multiple of it), the first two answers would have been the other way around.

Problem 3: Matrices

■ **Subproblem 3.1 [2 pts]:** Assume the following three matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

Answer the following questions:

1. Which of the three matrices is a diagonal matrix (if any)?
   Shortly explain your answer.
2. A is the inverse matrix of C. Is this correct?
   Shortly explain your answer.
3. C is the inverse matrix of A. Is this correct?
   Shortly explain your answer.
4. Use two of the three matrices to prove that matrix multiplication is not commutative.

■ **Solution/comments:**

1. In a diagonal matrix, all coefficients aside from the a_{ii} in the diagonal are zero. Since this is not the case for any of them, the correct answer is that none of them is a diagonal matrix.
2. The inverse of a matrix A is defined by $AA^{-1} = A^{-1}A = I$ (with I being the identity matrix). If we calculate AC and CA, we see that neither AC nor CA are equal to I. Hence, the statement is not correct.
3. From the explanation above it immediately follows that this statement is wrong, too.
4. We have to show that $M_1M_2 \neq M_2M_1$ for two matrices M_1 and M_2. Since none of the matrices are inverse to each other, we can take each random pair to prove this. For example, if we take A and B and calculate the resulting coefficient at position $(1,1)$ for AB, we get $1*1+1*2+1*3 = 6$. But for BA, we get $1*1+2*0+3*0 = 1$. So the resulting matrices are not the same, and hence matrix multiplication is not commutative.

■ **Subproblem 3.2 [1 pts]:** Prove that $(sA)^T = sA^T$ for all scalar values s and any matrix A.

■ **Solution/comments:**

Let's look at the left side of the equation first, i.e. $(sA)^T$. Let a_{ij} be the coefficient at a random position (i,j) of A and a_{ji} be the one at position (j,i). When we calculate sA, these values change to sa_{ij} and sa_{ji}, respectivley. Building the transposed of this resulting matrix sA means that the coefficients at positions (i,j) and (j,i) "switch places". So in the resulting matrix, the value for the coefficient at position (i,j) is $sa_{j,i}$, and the value for the coefficient at position (j,i) is $sa_{i,j}$.

Now let's look at the right side of the equation, i.e. sA^T. If we build the transposed of A, the values at positions (i,j) and (j,i) "switch places". So in the resulting matrix A^T, the value for the coefficient at position (i, j) is $a_{j,i}$, and the value for the coefficent at position (j,i) is $a_{i,j}$. If we multiply this matrix with the scalar s, these coefficients become $sa_{j,i}$ and $sa_{i,j}$, respectively.

We see that we end up with the very same coefficients. Because the position (i, j) was chosen randomly, this is true for any coefficient in the resulting matrix, which proves that $(sA)^T = sA^T$.

■ **Subproblem 3.3 [2 pts]:** Assume the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

1. Calculate the determinant of A.
2. Calculate the cofactor of a_{23}.

■ **Solution/comments:**

1. If we write two copies of the matrix next to each other, we can simply calculate the determinant by using the "trick" shown in the lecture (multiplying the coordinates on the diagonals, then adding up the ones for the diagonals from top right to bottom left, and subtracking the ones from the diagonals from top left to bottom right. With this we get

$$\det A = 1*2*1+2*1*1+3*3*0-0*1*1-1*3*2-1*2*3 = 2+2-6-6 = -8$$

2. For the cofactor a_{23}, we get

$$a_{23} = 1(-1)^{2+3} \det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = -1*-2 = 2$$

■ **Subproblem 3.4 [2 pts]:** Use Gaussian elimination to calculate the inverse A^{-1} of the following matrix:

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 4 & -2 & 4 \\ 2 & 4 & 2 \end{pmatrix}.$$

Note: Write down each step, so we can give you at least some credit even if your result is wrong due to some calculation errors.

■ **Solution/comments:**

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 4 & -2 & 4 & 0 & 1 & 0 \\ 2 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply 1st row with $1/2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 4 & -2 & 4 & 0 & 1 & 0 \\ 2 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply 1st row with -4 and -2 and add it to 2nd and 3rd row, respectively:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & -2 & -4 & -2 & 1 & 0 \\ 0 & 4 & -2 & -1 & 0 & 1 \end{array}\right)$$

Multiply 2nd row with $-1/2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1/2 & 0 \\ 0 & 4 & -2 & -1 & 0 & 1 \end{array}\right)$$

Multiply 2nd row with -4 and add it to 3rd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1/2 & 0 \\ 0 & 0 & -10 & -5 & 2 & 1 \end{array}\right)$$

Multiply 3rd row with $-1/10$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/5 & -1/10 \end{array}\right)$$

Multiply 3rd row with -2 and add it to 2nd and 1st row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 2/5 & 1/5 \\ 0 & 1 & 0 & 0 & -1/10 & 1/5 \\ 0 & 0 & 1 & 1/2 & -1/5 & -1/10 \end{array}\right)$$

Hence, the inverse of matrix A is

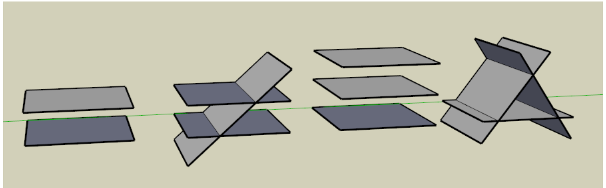
$$A^{-1} = \begin{pmatrix} -1/2 & 2/5 & 1/5 \\ 0 & -1/10 & 1/5 \\ 1/2 & -1/5 & -1/10 \end{pmatrix}$$

■ **Subproblem 3.5 [1 pts]:** The intersection of three planes in $\mathbb{R}^3$ can either be a single point, a line, or it is empty if the planes don't intersect. If we use Gaussian elimination to calculate the possible intersection, what happens in the latter case, i.e. an empty set of intersections? What is the geometric interpretation of this, i.e. how can the planes look like if their intersection set is empty?

■ **Solution/comments:** If there is no intersection, we end up getting one line such as $0x+0y+0z=d$ with some $d \neq 0$. This can not be solved, so there is no solution to the linear equation system, and hence no intersection.

If there is no intersection between the lines, there are basically two cases: either at least two planes are parallel to each other, or the intersections of every two pairs of planes (which are lines) are parallel to each other.

Note that alternatively to this verbal description, you could draw the following image:



(If you draw the image, you have to draw all cases. For the verbal description it is however sufficient to "summarize" the left three cases by one statement, i.e. at least two planes have to be parallel.)

Problem 4: Transformations

■ **Subproblem 4.1 [1.5 pts]:** T is a transformation matrix in 2D:

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

1. What kind of transformation does it realize?
2. Prove that it is a linear transformation.

■ **Solution/comments:** The matrix realizes a uniform scaling by a factor of 2 with respect to the origin.

T is a linear transformation if and only if $T(c_1\vec{u} + c_2\vec{v}) = c_1T(\vec{u}) + c_2T(\vec{v})$ for any vectors $\vec{u} = (x_u, y_u)$ and $\vec{v} = (x_v, y_v)$, and any scalars c_1 and c_2.

If we look at the concrete matrix T from this subproblem, we get for the left side of this equation:

$T(c_1\vec{u} + c_2\vec{v})$

$$= T(c_1 \begin{pmatrix} x_u \\ y_u \end{pmatrix} + c_2 \begin{pmatrix} x_v \\ y_v \end{pmatrix})$$

$$= T(\begin{pmatrix} c_1x_u \\ c_1y_u \end{pmatrix} + \begin{pmatrix} c_2x_v \\ c_2y_v \end{pmatrix})$$

$$= T(\begin{pmatrix} c_1x_u + c_2x_v \\ c_1y_u + c_2y_v \end{pmatrix})$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} c_1x_u + c_2x_v \\ c_1y_u + c_2y_v \end{pmatrix}$$

$$= \begin{pmatrix} 2c_1x_u + 2c_2x_v \\ 2c_1y_u + 2c_2y_v \end{pmatrix}$$

And for the right side we get:

$$c_1 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_u \\ y_u \end{pmatrix} + c_2 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_v \\ y_v \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 2x_u \\ 2y_u \end{pmatrix} + c_2 \begin{pmatrix} 2x_v \\ 2y_v \end{pmatrix}$$

$$= \begin{pmatrix} c_1 2x_u \\ c_1 2y_u \end{pmatrix} + \begin{pmatrix} c_2 2x_v \\ c_2 2y_v \end{pmatrix}$$

$$= \begin{pmatrix} c_1 2x_u + c_2 2x_v \\ c_1 2y_u + c_2 2y_v \end{pmatrix}$$

which is clearly the same as we have on the left side.

Of course, alternatively, one could prove that 1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and 2. $T(c\vec{u}) = cT(\vec{u})$

■ **Subproblem 4.2 [2 pts]:** In this subproblem, we look at transformations in 2D.

1. Give a matrix for a counterclockwise rotation around the origin about a random angle ϕ.
2. Give a matrix for a translation from point $(2,2)$ to the origin.
3. Give a matrix for a counterclockwise rotation around point $(2,2)$ about a random angle ϕ. Shortly explain how you got your answer.

■ **Solution/comments:**

1. $T = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

   Note: adding homogeneous coordinates is not necessary here, but it's not wrong either, so both solutions are correct.
2. $T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

   Note: here it is obviously necessary to add homogeneous coordinates.
3. We can do this by translating everything from $(2,2)$ to the origin, do the rotation around the origin, and then translate everything back with the inverse operation of the first translation. We already have the first two matrices. The third one is just the inverse of the first one, so we get

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & -2\cos\phi + 2\sin\phi + 2 \\ \sin\phi & \cos\phi & -2\sin\phi - 2\cos\phi + 2 \\ 0 & 0 & 1 \end{pmatrix}$$

■ **Subproblem 4.3 [2 pts]:** Look at the following transformation matrix T:

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. If we apply this transformation matrix to an object in 3D, what happens to the object?
2. Modify this matrix, so it does not only apply this operation but also doubles the size of the object with respect to the origin.

■ **Solution/comments:**

1. Each point $(x,y,z,1)$ becomes $(-x,-y,-z,1)$, so the matrix realizes point reflection at the origin, i.e. the object gets reflected at the origin.

2. The additional operation is simply the scaling matrix from the first subproblem generalized to 3D, i.e.

$$T_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combining this with the point reflection give us

$$T_1 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that in this particular case, it actually doesn't matter in which order we apply the operations. But this is not always true!

■ **Subproblem 4.4 [1.5 pts]:** Look a the following transformation matrix T:

$$T = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

1. And what happens if we apply it to a vector $\vec{v} = (x, y, z)$ in **3D**?
2. What happens if we apply this transformation matrix T to a location (x, y) in **2D**
3. In the previous case, we used a 3×3 matrix to do a transformation in 2D. Explain why.

   Note: In order to get full credits, it is sufficient to say how the coefficients in the last row are called and why we need them.

■ **Solution/comments:**

1. In 3D, the matrix realizes a shearing in Z-direction, so the vector $\vec{v} = (x, y, z)$ becomes the vector $\vec{v}^* = (x + az, y + bz, z)$
2. In 2D, the matrix realizes a translation. Because we need homogeneous coordinates for this, the location is represented as $(x, y, 1)$. Applying the matrix to it, give us the new location $(x + a, y + b, 1)$.
3. Translation is not a linear transformation. Hence, we can't do it with regular matrix multiplication in 2D. Instead, we have to use homogeneous coordinates, which are created by adding the last row to the matrix and a 1 or a 0 to locations and vectors, respectively.