Graphics (INFOGR 2011-2012): Midterm Exam (T1)

Friday, May 25, 2012, EDUC-GAMMA, 09:00-12:00 (duration: 2 hours)

COMMENTS AND SOLUTIONS

No responsibility is taken for the correctness of the provided information.

Note that there can be more than one correct solution for most problems, which is why the following information should not necessarily be considered as standard solution to the problems.

Problem 1: Vectors

■ Subproblem 1.1 [8 pts]: Multiple Choice Questions

1. The two vectors $(1,2)$ and $(-2,1)$ are …

   A. … linearly dependent of each other.

   B. … forming an orthonormal basis.

   C. … perpendicular to each other.

   D. … pointing in the opposite direction of each other.

   ■ **Solution/comments:** The correct answer is C: "… are perpendicular to each other."

   Explanation (not required): Their scalar product is 0, which is why they must be perpendicular to each other. This also excludes the first and last option. To be an orthonormal basis, they also have to be unit vectors. But their length is $\sqrt{1+4} \neq 1$, so this is not the case.

2. If s is a scalar value and $\vec{v}$, $\vec{w}$ are two vectors in 3D, then the result of $s + (\vec{v} \times \vec{w}) \cdot (\vec{w} \times \vec{v})$ is …

   A. … a vector in 3D.

   B. … undefined.

   C. … a scalar.

   D. … a 3x3 matrix.

   Note: "$\times$" denotes the cross product, "$\cdot$" denotes the scalar product (also called dot product or inner product).

   ■ **Solution/comments:** The correct answer is C: "… a scalar."

   Explanation (not required): The cross product $\times$ returns a vector. The scalar product of two vectors returns a scalar (hence the name "scalar" product). Adding a scalar to another one delivers again a scalar value.

3. Let θ be the angle between two vectors (neither of which is the null vector). If $90° < \theta < 270°$, the scalar product of these two vectors is …

   A. … positive.

   B. … negative.

   C. … undefined.

   D. … positive for $\theta < 180°$ and negative for $\theta > 180°$.

   ■ **Solution/comments:** The correct answer is B: "… negative."

   Explanation (not required): We have $\vec{v} \cdot \vec{w} = \cos\theta \cdot \|\vec{v}\| \cdot \|\vec{w}\|$. If $90° < \theta < 270°$, $\cos\theta < 0$. Because the length of a vector is always positive, the scalar product is negative.

4. If the scalar product of two unit vectors is zero, they are …

   A. … linearly dependent.

   B. … forming an orthonormal basis.

   C. … at an angle of 45° or 225° to each other.

   D. … pointing in the same direction.

   ■ **Solution/comments:** The correct answer is B: "… forming an orthonormal basis."

   Explanation (not required): If $\vec{v} \cdot \vec{w} = \cos\theta \cdot \|\vec{v}\| \cdot \|\vec{w}\| = 0$ and neither of the vectors is the null vector, they must be perpendicular to each other. This already excludes the first, third, and last answer. The condition for being an orthonormal basis is that the angle between them is 90°, and each of them must have a length of 1 (which in turn is how a unit vector is defined).

■ Subproblem 1.2 [6 pts]: Vector calculation

Assume two vectors $\vec{u}$, $\vec{v}$ in 3D with $\vec{u} = (2,2,1)$ and $\vec{v} = (4,4,1)$.

1. Calculate the scalar product $\vec{u} \cdot \vec{v}$. (Note: the scalar product is also called dot product or inner product.)

   ■ **Solution/comments:** $\vec{u} \cdot \vec{v} = 2 \cdot 4 + 2 \cdot 4 + 1 \cdot 1 = 17$

2. Create a unit vector $\vec{u'}$ that points in the same direction as vector $\vec{u}$.

   ■ **Solution/comments:**

   Length of vector $\vec{u}$: $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{2^2 + 2^2 + 1^2} = 3$

   $\curvearrowright \vec{u'} = \frac{\vec{u}}{\|\vec{u}\|} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

3. Are the two vectors $\vec{u}$ and $\vec{v}$ linearly dependent or linearly independent? Shortly justify your answer.

   ■ **Solution/comments:** If they are linarly dependent, there must be a scalar $\lambda \neq 0$ with $(u_1, u_2, u_3) = \lambda(v_1, v_2, v_3)$. In this particular case we have:

For the 1st coordinate:	$2 = \lambda\, 4$
For the 2nd coordinate:	$2 = \lambda\, 4$
For the 3rd coordinate:	$1 = \lambda\, 1$

   The first two equations are only fulfilled for $\lambda = 2$, the last one only for $\lambda = 1$. Hence, a single, unique λ does not exist. The two vectors are **linearly independent**.

■ Subproblem 1.3 [6 pts]: Vector characteristics

Assume two vectors $\vec{w}$, $\vec{v}$ in 3D with $\vec{w} = \lambda\vec{v}$ and $\lambda = \frac{1}{2}$.
Prove that vector $\vec{w}$ is half as long as vector $\vec{v}$.

■ **Solution/comments:** We have $\vec{w} = (w_1, w_2, w_3) = (\lambda v_1, \lambda v_2, \lambda v_3) = (\frac{v_1}{2}, \frac{v_2}{2}, \frac{v_3}{2})$, and want to show that the length of $\vec{w}$ is half of the length of $\vec{v}$, i.e. that $\|\vec{w}\| = \frac{1}{2}\|\vec{v}\|$.

$$\|\vec{w}\| = \sqrt{(\frac{v_1}{2})^2 + (\frac{v_2}{2})^2 + (\frac{v_3}{2})^2} = \sqrt{\frac{v_1^2 + v_2^2 + v3^2}{2^2}} = \frac{1}{2}\sqrt{v_1^2 + v_2^2 + v_3^2} = \frac{1}{2}\|\vec{v}\|$$

q.e.d.

Problem 2: Basic Geometric Entities

■ Subproblem 2.1 [10 pts]: Multiple Choice Questions

1. The equation $2x+y+z-10=0$ represents ...
   A. ... the implicit representation of a line in 3D.
   B. ... the parametric representation of a line in 3D.
   C. ... the slope-intercept representation of a line in 3D.
   D. ... the implicit representation of a plane in 3D.
   E. ... the parametric representation of a plane in 3D.
   F. ... the slope-intercept representation of a plane in 3D.
   G. none of the above

   ■ **Solution/comments:** The correct answer is D.

2. Which of the following vectors are normal vectors of the vector (x,y) in 2D $(x,y \neq 0)$?
   A. $(-x,y)$
   B. $(x,-y)$
   C. $(-x,-y)$
   D. (y,x)
   E. $(y,-x)$
   F. $(-y,x)$
   G. $(-y,-x)$
   H. none of the above

   ■ **Solution/comments:** The correct answers are E and F.

3. In a 3-dimensional space the equation $(x-2)^2+(y-2)^2+(z-2)^2-4=0$ represents ...
   A. ... an implicit representation of a sphere with radius 2 and center $(2,2,2)$.
   B. ... an implicit representation of a sphere with radius 2 and center $(-2,-2,-2)$.
   C. ... an implicit representation of a sphere with radius 4 and center $(2,2,2)$.
   D. ... an implicit representation of a sphere with radius 4 and center $(-2,-2,-2)$.
   E. ... a parametric representation of a sphere with radius 2 and center $(2,2,2)$.
   F. ... a parametric representation of a sphere with radius 2 and center $(-2,-2,-2)$.
   G. ... a parametric representation of a sphere with radius 4 and center $(2,2,2)$.
   H. ... a parametric representation of a sphere with radius 4 and center $(-2,-2,-2)$.
   I. none of the above

   ■ **Solution/comments:** The correct answer is A.

4. The intersection of three planes in 3D can be ... ?
   A. ... empty.
   B. ... a point.
   C. ... a line.
   D. ... a plane.
   E. ... a parallelogram.
   F. ... three lines that are parallel to each other.
   G. none of the above

■ **Solution/comments:** There was a mistake in the problem description: it should have been "the intersection of ALL three planes", in which case the correct answers would be A, B, C, and D. Not F, because the three lines do not intersect each other, so they are not an intersection with ALL planes. However, since we forgot to put the "all" in the problem description, both solutions, i.e. A, B, C, D as well as A, B, C, D, F would have given you full credit.

5. The following equation represents a plane in 3D:

$$\begin{pmatrix}1\\1\\1\end{pmatrix}(\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}0\\0\\3\end{pmatrix})=0$$

Which of the following statements are correct?

A. It is an implicit representation.
B. It is a slope-intercept representation.
C. It is a parametric representation.
D. It is parallel to the vector $(1,1,1)$.
E. It intersects with the $x-$axis at $(0,0,3)$.
F. It intersects with the $z-$axis at $(0,0,3)$.
G. It contains the point $(2,-1,1)$.
H. It contains the point $(2,-1,2)$.
I. none of the above

■ **Solution/comments:** The correct answers are A, F, H.

■ Subproblem 2.2 [12 pts]: Planes in 3D

Assume the vectors $\vec{p}_0=(1,1,1)$, $\vec{p}_1=(2,3,3)$, and $\vec{p}_2=(3,1,1)$ that are representing three points in 3D.

1. Write down the parametric representation of a plane that is defined by these three points. Use $\vec{p}_0$ as support vector in your equation.

   ■ **Solution/comments:** General form of a parametric equation of a plane in 3D:

$$\vec{p}(s,t)=\vec{p}_0+s(\vec{p}_1-\vec{p}_0)+t(\vec{p}_2-\vec{p}_1)$$

   Hence, we get

$$\vec{p}(s,t)=\begin{pmatrix}1\\1\\1\end{pmatrix}+s\begin{pmatrix}1\\2\\2\end{pmatrix}+t\begin{pmatrix}2\\0\\0\end{pmatrix}$$

   Notice that any scalar multiple of the direction vectors above is correct, too. Also, if you used a different (but correct) support vector, you only got a minor decrease and almost full credit.

2. Give a normal vector $\vec{n}$ to the plane created above.

   ■ **Solution/comments:** We can get this by taking the cross product of two vectors on the plane, e.g. the two direction vectors from the parametric equation:

$$\vec{n}=\begin{pmatrix}1\\2\\2\end{pmatrix}\times\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}0\\4\\-4\end{pmatrix}$$

3. Give the implicit representation of the above plane in vector notation.

   ■ **Solution/comments:** The implicit representation in vector form is $\vec{n}(\vec{p}-\vec{p}_0)=0$. Because we already calculated the normal vector for the previous question, we can just write it down:

$$\begin{pmatrix}0\\4\\-4\end{pmatrix}(\vec{p}-\begin{pmatrix}1\\1\\1\end{pmatrix})=0$$

4. Give the parametric representation of a plane in 3D that contains the point $(1,2,2)$ and is parallel to the following two lines:

$$l_1(s) = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix} + s \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} \text{ and } l_2(t) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

■ **Solution/comments:** There are different correct solutions, but the most obvious is:

$$\vec{p}(t) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + s \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

Explanation (not required): If a line is parallel to a plane, all vectors on this line must be parallel to all vectors on the plane. Therefore, the direction vector of the line must be a direction vector of the plane, too. Because the direction vectors of $l_1(s)$ and $l_2(t)$ are not linearly dependent, we can use them as the two direction vectors for the plane. Using the point $(1,2,2)$ as support vector, we get the equation above.

■ Subproblem 2.3 [8 pts]: Barycentric coordinates

Assume the following three vectors that represent a triangle in 3D:

$$\vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}, \vec{c} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

1. Write down how we would represent a point in the barycentric coordinate system defined by this triangle.

■ **Solution/comments:** The general form of barycentric coordinates is:

$$\vec{p}(\beta,\gamma) = \vec{a} + \beta(\vec{b} - \vec{a}) + \gamma(\vec{c} - \vec{a})$$

In this particular case, we get:

$$\vec{p}(\beta,\gamma) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

Notice that other correct solutions exist depending on what vector you pick to represent $\vec{a}$ in the general form. Also, you got full credit if you wrote it the slightly different form used in the book.

2. What are the conditions that have to be fulfilled if a point that is expressed in barycentric coordinates is within this triangle (including its edges and vertices)?

■ **Solution/comments:**

$$\beta, \gamma \geq 0 \text{ and } \beta + \gamma \leq 1$$

3. Verify if the point $(2,3,3)$ is inside the triangle or not using the conditions from the previous question.

■ **Solution/comments:** We have:

$$\vec{p}(\beta,\gamma) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

The first line gives us $4\beta + 2\gamma = 2 \curvearrowright \gamma = 1 - 2\beta$.

Putting this into the second line, i.e.$4\beta + 4\gamma = 3$ gives us $4\beta + 4(1 - 2\beta) = 3 \curvearrowright \beta = \frac{1}{4}$.

Using β in any of the lines, e.g. the first one, gives us $4\beta + 2\gamma = 2 \curvearrowright 1 + 2\gamma = 2 \curvearrowright \gamma = \frac{1}{2}$

Because β and γ are both ≥ 0 and $\beta + \gamma = \frac{3}{4} \leq 1$, the point is within the triangle.

Problem 3: Matrices and Determinants

■ Subproblem 3.1 [7 pts]: Multiple Choice Questions

1. Let v be a vector and A is a 3×3 matrix, B is a 2×3 matrix, C is a 3×2 matrix, D is a 3×3 matrix. Which of the following calculations are defined?

   A. AB

   B. BA

   C. $ABCv$

   D. $CBAv$

   E. $ABCD$

   F. $DCBA$

   G. none of the above

   ■ Solution/comments:

   Answers B, D, and F are correct if v is a vector in 3D.
   Answers B, F are correct if v is a vector in 2D.
   Since we forgot to specify it, both solutions gave full credit.

2. Which of the following statements are correct, if $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$.

   A. $\det A = 3$

   B. $\det A = -3$

   C. $\det A^T = 3$

   D. $\det A^T = -3$

   E. none of the above

   ■ Solution/comments: Answers A and C are correct.

3. Which of the following statements are correct, if $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   A. A is the identity matrix.

   B. A is a diagonal matrix.

   C. A is an orthogonal matrix.

   D. $A = A^{-1}$

   E. $A = B^T$

   F. $A^T + B$ is a diagonal matrix.

   G. none of the above

   ■ Solution/comments: Answers C and D are correct.

■ Subproblem 3.2 [6 pts]: Matrix arithmetics

- Calculate the result of the following matrix multiplication:

$$\begin{pmatrix} 3 & 1 \\ 4 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -5 & 2 \end{pmatrix}$$

  ■ Solution/comments: The solution is

$$\begin{pmatrix} 4 & 17 \\ 12 & 20 \\ -11 & -8 \end{pmatrix}$$

- In the tutorials, there was one exercise where we have proven that:

  *If $AB = AC$, it does **not** necessarily follow that $B = C$.*

  However, by using matrix arithmetics, we can get:

$$\begin{array}{rcll} AB & = & AC & \Rightarrow \\ A^{-1}AB & = & A^{-1}AC & \Rightarrow \\ IB & = & IC & \Rightarrow \\ B & = & C & \end{array}$$

  Explain what is wrong here. (Hint: what we have proven in the tutorials is correct. One short sentence should be enough to explain the issue.)

  ■ **Solution/comments:** The second statement is correct if and only if the inverse matrix A^{-1} is defined, which is not always the case.

■ Subproblem 3.3 [7 pts]: Determinants

1. Compute the determinant of the matrix: $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

   ■ **Solution/comments:** The solution is 8.

2. Compute the cofactor a^c_{23} of the matrix A above. (Hint: cofactors are mostly used to do Laplace's expansion. You do **not** have to do the full expansion here, but just have to calculate this one single cofactor.)

   ■ **Solution/comments:** For the cofactor of the coefficient a_{23} we get:

$$a^c_{23} = 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}(-1)^{2+3} = 1 \cdot 2 \cdot (-1) = -2$$

3. Assume two vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ that are the column vectors of the matrices $M_1 = (\vec{a}\vec{b})$ and $M_2 = (\vec{b}\vec{a})$. Prove that $\det M_1 = -\det M_2$, i.e. that $|\vec{a}\vec{b}| = -|\vec{b}\vec{a}|$.

   ■ **Solution/comments:**

   For M_1, we get $\det M_1 = a_1 b_2 - b_1 a_2$.

   For M_2, we get $\det M_2 = b_1 a_2 - a_1 b_2$.

   So, we have $-\det M_2 = -b_1 a_2 + a_1 b_2$ which in indeed $\det M_1$

   q.e.d.

■ Subproblem 3.4 [5 pts]: LESs and Gaussian Elimination

Assume the following linear equation system (LES):

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 4y & + & 7z & = & 16 \\ 2x & + & 2y & + & 4z & = & 10 \end{array}$$

1. What is the geometric interpretation of this LES?
2. Create the augmented matrix and solve it using Gaussian Elimination.
3. What is the geometric interpretation of your solution?

■ **Solution/comments:**

- The LES represents three planes in 3D.
- Gaussian elimination:

  The augmented matrix is:

  $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 4 & 7 & 16 \\ 2 & 2 & 4 & 10 \end{array}\right)$$

  Multiply 1st row with -1 and -2 and add it to the 2nd and 3rd row, respectively:

  $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

  Multiply 2nd row with $\frac{1}{3}$:

  $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

  Multiply 2nd row with -1 and add it to 1st row:

  $$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

  Multiply 3rd row with $\frac{1}{2}$:

  $$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

  Multiply 3rd row with -2 and 1 and add it to the 1st and 2nd row, respectively:

  $$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

  The solution is $x = 1$, $y = 2$, and $z = 1$.
- The solution represents the intersection point of all three planes.

Problem 4: Transformations

■ Subproblem 4.1 [5 pts]: Multiple Choice Questions

1. Assume the following transformation matrix $T = \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   The values in the third column represent . . .

   A. . . . the image of the 2nd base vector under the linear transformation
   B. . . . the image of the 2nd base vector under the affine transformation
   C. . . . the image of the origin under the linear transformation
   D. . . . the image of the origin under the affine transformation
   E. . . . none of the above

   The values in the second column represent . . .

   A. . . . the image of the 2nd base vector under the linear transformation
   B. . . . the image of the 2nd base vector under the affine transformation
   C. . . . the image of the origin under the linear transformation
   D. . . . the image of the origin under the affine transformation
   E. . . . none of the above

   The values in the first row represent . . .

   A. . . . the image of the 2nd base vector under the linear transformation
   B. . . . the image of the 2nd base vector under the affine transformation
   C. . . . the image of the origin under the linear transformation
   D. . . . the image of the origin under the affine transformation
   E. . . . none of the above

   ■ Solution/comments:

   The correct answers are

   1. E for the 1st question
   2. A for the 2nd question
   3. E for the 3rd question

2. The matrix $T = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ can realize a counterclockwise rotation about the angle ϕ . . .

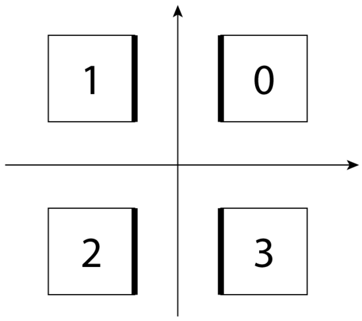
   A. . . . around the origin in 2D.
   B. . . . around the $x-$axis in 2D.
   C. . . . around the $y-$axis in 2D.
   D. . . . around the $z-$axis in 2D.
   E. . . . around the origin in 3D.
   F. . . . around the $x-$axis in 3D.
   G. . . . around the $y-$axis in 3D.
   H. . . . around the $z-$axis in 3D.
   I. none of the above

   ■ Solution/comments:

   Answers A and H are correct.

■ Subproblem 4.2 [10 pts]: Finding Transformation Matrices

1. In the following image, the object on the top right (marked with "0") is transformed by three different transformation matrices T_1, T_2, and T_3. The result of these transformations are illustrated by the three objects marked with "1", "2", and "3", respectively. (Notice that the numbers are just there to mark the object but not part of the actual objects and transformations.)



   Write down the three transformation matrices.

   ■ Solution/comments:

$$T_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   Explanation (not required): If we look at one edge of the square (let's call it the point (x,y)), and check where it is translated to, we see that for T_1 it is mapped to $(-x,y)$, for T_2 it is mapped to $(-x,-y)$, and for T_3 it is mapped to $(x,-y)$. Getting the matrices that do such a transformation is straightforward. Alternatively, we could have said that T_1 and T_3 are realizing a reflection on one of the axes and T_2 realizes a point reflection at the origin. Again, looking how such a transformation changes the $x-$ and $y-$values gives us our three matrices.

   Note: Some people misinterpreted the problem description and provided three transformation matrices that would turn object 0 to object 1, then object 1 to object 2, then object 2 to object 3 (instead of turning object 0 into each of the other 3). The problem clearly states otherwise, but since I don't want anyone to have any disadvantages because this course is not given in their native language (and I assume such misunderstandings would have been less likely if it were written in Dutch), we were generous and gave full credit for this solution as well.

2. Give a matrix for a linear transformation that maps the point $(1,0)$ to $(8,1)$ and the point $(0,1)$ to $(7,3)$.

   ■ Solution/comments:

$$M = \begin{pmatrix} 8 & 7 \\ 1 & 3 \end{pmatrix}$$

   Explanation (not required): The trick here is to recognize that the two vectors that we want to transform are exactly the base vectors. Because the columns of our transformation matrix are exactly the images of these vectors after the transformation, we can just write them down without even thinking about what kind of transformation this might be.

■ Subproblem 4.3 [10 pts]: Linear and Affine Transformations

1. Give the matrix for uniform scaling with respect to the origin by a factor of 2 in 2D.

   ■ Solution/comments:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

2. Give the matrix for uniform scaling with respect to the point $(1,1)$ by a factor of 2 in 2D.

   ■ **Solution/comments:**

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

   Explanation (not required): The 1st and 2nd column are the images of the base vectors under the linear transformation (scaling), so the same as in the previous question. The right column is the the image of the origin under the affine transformation (scaling and translation). If we scale the vector from our scaling center $(1,1)$ to the origin $(0,0)$ by a factor of two, we end up at $(-1,-1)$.

   Alternatively, one could have solved this problem by saying we first move everything by a vector from $(1,1)$ to $(0,0)$ (i.e. from the scaling center to the origin), then do our scaling, and then move everything back by the reversed vector. If you create the three matrices for this operation and multiply them in the right order, we get the same result as above:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Give the matrix for nonuniform scaling with respect to the point $(1,1)$ by a factor of 2 in $x-$direction and a factor of 4 in $y-$direction in 2D.

   ■ **Solution/comments:**

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

   Explanation (not required): Like above, the first two column vectors should be obvious. The last one is again the image of the origin under the affine transformation. If we scale the vector from $(1,1)$ to $(0,0)$ by a factor of 2 and 4 in $x-$ and $y-$direction respectively, we end up at $(-1,-3)$.

   And again, there is also an alternative way to get this solution:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

   Notice that the matrices for the translation are of course the same because we are still moving everything from our desired scaling center to the origin. But because the operation done there, the image of the origin is not the same, hence, the values in the right column of our final matrix are different.