Graphics (INFOGR 2012-2013): Midterm Exam (T1)

Thursday, May 30, 2013, EDUC-GAMMA, 13:30-16:30 (time for the exam: max. 2 hours)

COMMENTS AND SOLUTIONS

No responsibility is taken for the correctness of the provided information.

Note: the provided solution might not be the only option,
but for some problems others might exist that are correct as well.

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

Problem 1: Vectors

■ **Subproblem 1.1 [8 pts]: Multiple choice questions.** Mark the correct answer. No explanation required. There is only one correct answer for each individual question.

1. The two vectors $(3,2,1)$ and $(6,4,2)$ are ...
   - A. parallel
   - B. linearly independent
   - C. unit vectors
   - D. an orthonormal basis
   - E. neither of these

   ■ **Solution/comments:** The correct answer is *A. parallel*.

2. The scalar product (aka dot product or inner product) of two perpendicular vectors is ...
   - A. -2π
   - B. -1
   - C. 0
   - D. 1
   - E. 2π
   - F. neither of these

   ■ **Solution/comments:** The correct answer is *C. 0*.

3. The scalar product of two vectors that form an angle of 270 degree with each other is ...
   - A. -2π
   - B. -1
   - C. 0
   - D. 1
   - E. 2π
   - F. neither of these

   ■ **Solution/comments:** The correct answer is *C. 0*.

4. In the following, $\cdot$ denotes scalar multiplication of two vectors and $\times$ denotes their cross product. If $\vec{v}, \vec{w}$ are two 3D vectors and s, t are two scalar values, then $(s\vec{v} \times t\vec{w}) + (s\vec{v} \cdot t\vec{w})$ is ...
   - A. undefined
   - B. a scalar value
   - C. a 2D vector
   - D. a 3D vector
   - E. a unit vector
   - F. neither of these

   ■ **Solution/comments:** The correct answer is *A. undefined*.

■ **Subproblem 1.2 [7 pts]: Simple vector calculations.**

1. What is the Euclidean length $||\vec{v}||$ of the vector $\vec{v} = (0,3,4)$?

   ■ **Solution/comments:** $||\vec{v}|| = \sqrt{0^2+3^2+4^2} = \sqrt{0+9+16} = \sqrt{25} = 5$

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

2. What is the scalar product $\vec{v} \cdot \vec{w}$ of the two vectors $\vec{v} = (1,2,3)$ and $\vec{w} = (2,2,2)$?

■ **Solution/comments:** $\vec{v} \cdot \vec{w} = 1 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 = 2 + 4 + 6 = 12$

3. What is the cross product $\vec{v} \times \vec{w}$ of the two vectors $\vec{v} = (1,2,3)$ and $\vec{w} = (4,1,1)$?

■ **Solution/comments:** $\vec{v} \times \vec{w} = \begin{pmatrix} 2 \cdot 1 - 3 \cdot 1 \\ 3 \cdot 4 - 1 \cdot 1 \\ 1 \cdot 1 - 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 12-1 \\ 1-8 \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \\ -7 \end{pmatrix}$

■ **Subproblem 1.3 [5 pts]: Vector characteristics.**

Prove that $\lambda(-y,x)$ is a normal vector to (x,y) for all $\lambda \neq 0$.

■ **Solution/comments:**

Scalar product: $\lambda(-y,x) \cdot (x,y) = (-\lambda y, \lambda x) \cdot (x,y) = -\lambda yx + \lambda xy = 0$

q.e.d.

Explanation (not required):
If the scalar product of two vectors is 0, then they are perpendicular to each other.
A vector that is perpendicular to another one is called normal vector to that vector.
Hence, showing that the scalar product is indeed 0 in this case proves the statement.

Problem 2: Basic geometric entities

■ **Subproblem 2.1 [6 pts]: Multiple choice questions.** Mark the correct answer. No explanation required. There is only one correct answer for each individual question.

1. If $y = ax + c$ denotes the *slope-intercept form* of a line in 2D, then c gives us ...
   A. the slope of the line
   B. the fraction of the slope in $X-$direction
   C. the fraction of the slope in $Y-$direction
   D. the intersection of the line with the $X-$axis
   E. the intersection of the line with the $Y-$axis
   F. neither of these

   ■ **Solution/comments:** The correct answer is *E. the intersection of the line with the $Y-$axis.*

2. If $2x - 3y + 5 = 0$ denotes the *implicit representation* of a line in 2D, then the vector $(2, -3)$ is ...
   A. a vector on the line
   B. a vector pointing to the line
   C. a vector perpendicular to the line
   D. a vector parallel to the line
   E. neither of these

   ■ **Solution/comments:** The correct answer is *C. a vector perpendicular to the line.*

3. If $\vec{p}(t) = (2,3) + t(4,1)$ denotes the *parametric equation* of a line in 2D, then the vector $(2,3)$ is ...
   A. the line's support vector
   B. the line's direction vector
   C. a normal to the line
   D. a vector parallel to the line
   E. neither of these

   ■ **Solution/comments:** The correct answer is *A. the line's support vector.*

■ **Subproblem 2.2 [8 pts]: Basic geometric entities in 3D.**

1. Write down the *parametric equation* of a plane in 3D that goes through the three points represented by the vectors $\vec{p}_0 = (1,2,3)$, $\vec{p}_1 = (3,3,3)$, and $\vec{p}_2 = (3,2,3)$. Use $\vec{p}_0$ as support vector of the plane.

   ■ **Solution/comments:**

$$p\vec{(t)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

   Comment: For the direction vectors, any scalar multiple of the following vectors would have been correct, too:

$$\vec{p}_1 - \vec{p0} = (2,1,0)$$
$$\vec{p}_2 - \vec{p0} = (2,0,0)$$
$$\vec{p}_1 - \vec{p2} = (0,1,0)$$

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

2. Write down the *implicit equation* of a plane in 3D in vector notation, i.e. $\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$ that contains the point $(1,1,1)$ and is parallel to the two vectors $\vec{v} = (1,2,3)$ and $\vec{w} = (4,1,1)$. (Hint: it is allowed to reuse results from previous problems.)

   Solution/comments:

$$\begin{pmatrix} -1 \\ 11 \\ -7 \end{pmatrix} \cdot (\vec{p} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) = 0$$

   Comment: Any scalar multiple of $\vec{n} = (-1, 11, -7)$ would have been correct, too.

3. The following represents the *parametric equation* of a sphere around the origin in 3D:

$$\vec{p}(\Phi, \Theta) = \begin{pmatrix} 5\cos\Phi\sin\Theta \\ 5\sin\Phi\sin\Theta \\ 5\cos\Theta \end{pmatrix}$$

   Write down the parametric equation of a sphere around the point $\vec{c} = (2,3,1)$ with the same radius.

   Solution/comments:

$$\vec{p}(\Phi, \Theta) = \begin{pmatrix} 2 + 5\cos\Phi\sin\Theta \\ 3 + 5\sin\Phi\sin\Theta \\ 1 + 5\cos\Theta \end{pmatrix}$$

4. The following represents the *implicit equation* of a sphere around the origin in 3D:

$$x^2 + y^2 + z^2 - 9 = 0$$

   Write down the implicit equation of a sphere around the point $\vec{c} = (2,3,1)$ with the same radius.

   Solution/comments:

$$(x-2)^2 + (y-3)^2 + (z-1)^2 - 9 = 0$$

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

Problem 3: Intersections

■ **Subproblem 3.1 [6 pts]: Multiple choice questions.** No explanation required.

1. When calculating the intersection of two lines in 3D, which of the following can be a possible outcome? Multiple answers might be correct. You only get credit for this subquestion if you mark *all* correct ones and none that is incorrect.

   A. no solution B. a point C. a line D. a plane E. a circle F. a triangle

   ■ **Solution/comments:** The correct answers are *A. no solution, B. a point, C. a line*.

2. When calculating the intersection of two spheres in 3D, which of the following can be a possible outcome? Multiple answers might be correct. You only get credit for this subquestion if you mark *all* correct ones and none that is incorrect.

   A. no solution B. a point C. a line D. a plane E. a circle F. a triangle

   ■ **Solution/comments:** The correct answers are *A. no solution, B. a point, E. a circle*.

3. Assume a point in 3D in *barycentric coordinates*, i.e. $\vec{p}(\beta,\gamma) = \vec{a} + \beta(\vec{b} - \vec{a}) + \gamma(\vec{c} - \vec{a})$ with $\beta \geq 0$ and $\gamma \geq 0$. What other condition has to be fulfilled if the point is within the triangle defined by the three points $\vec{a}$, $\vec{b}$, $\vec{c}$? There is only one correct answer to this question.

   A. $\beta \leq 1, \gamma \leq 1$
   B. $\beta + \gamma \leq 1$
   C. $\beta + \gamma = 1$
   D. $\frac{1}{2}(\beta + \gamma) \leq 1$
   E. $\frac{1}{2}(\beta + \gamma) = 1$
   F. neither of these

   ■ **Solution/comments:** The correct answer is *B.* $\beta + \gamma \leq 1$.

■ **Subproblem 3.2 [6 pts]: Intersection of lines.**

Assume two lines in 2D: line l_1 is defined by the two points $(1,2)$ and $(3,3)$ and line l_2 is defined by the two points $(1,1)$ and $(3,2)$. Do these two lines intersect? If yes, calculate the intersection point(s). If no, explain why.

■ **Solution/comments:**

No, these two lines do not intersect.

If we calculate their direction vectors, we get $\vec{d_1} = (2,1)$ for l_1 and $\vec{d_2} = (2,1)$ for l_2, so they are either parallel or identical. Because each support vector is not part of the other line (no matter which one we chose as support vector), they cannot be identical. Hence they are parallel and share no intersection points.

Comment: Obviously, there are other ways to prove this. For example, we could have just created the line equations and tried to calculate the intersection, which would have led to an unsolvable linear equation system.

Problem 4: Matrices and determinants

■ **Subproblem 4.1 [6 pts]: Multiple choice questions.** No explanation required.
There is only one correct answer for each individual question.

1. Which of the following statements is correct? If A is a diagonal matrix, then ...
   A. $A^T = I$
   B. $A^T = A$
   C. $A^T = A^{-1}$
   D. $A^{-1} = I$
   E. A^{-1} is always undefined
   F. neither of these

   ■ **Solution/comments:** The correct answer is *B.* $A^T = A$.

2. Which of the following statements is **not** true for any three random matrices A, B, and C with the same dimensions?
   A. $(AB)C = A(BC)$
   B. $AB = BA$
   C. $A(B+C) = AB + AC$
   D. $(A+B)C = AC + BC$

   ■ **Solution/comments:** The correct answer is *B.* $AB = BA$.

3. Assume we have a 2×3 matrix A, a 2D vector $\vec{v}$, a 3D vector $\vec{w}$, and $\cdot$ denotes matrix multiplication.
   What is the result of $(\vec{v} \cdot A \cdot \vec{w}^T) + (\vec{w} \cdot A^T \cdot \vec{v}^T)$?
   A. a 3×1 matrix
   B. a 2×1 matrix
   C. a 1×1 matrix
   D. a 1×2 matrix
   E. a 1×3 matrix
   F. neither of those

   ■ **Solution/comments:** The correct answer is *F. neither of those.*

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

■ **Subproblem 4.2 [4 pts]: Simple matrix calculations.**

Assume the following two matrices: $A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 1 & 2 \end{pmatrix}$.

1. Calculate the product AB using matrix multiplication.

   ■ **Solution/comments:** $AB = \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix}$

2. Calculate the sum $A + B^T$.

   ■ **Solution/comments:** $A + B^T = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 3 & 3 \end{pmatrix}$

■ **Subproblem 4.3 [5 pts]: Matrix characteristics.**

Assume a 2×2 matrix A where the two column vectors are parallel to each other.
Prove that the determinant $\det A$ of this matrix is zero.

■ **Solution/comments:**

The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

If the column vectors of this matrix A are parallel to each other, there must be a λ such that $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda c \\ \lambda d \end{pmatrix}$.

Hence, we get

$$\det A = \lambda cd - \lambda dc = 0$$

q.e.d.

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

■ **Subproblem 4.4 [8 pts]: Determinants.**

1. Complete the following sentence in a way that it becomes a correct statement.

   The determinant of a 2×2 matrix is the _____*oriented*_____*area*

   of the _____*parallelogram*_____*defined by its two column vectors.*

   Note: *signed* volume instead of oriented area would be correct, too.

2. Calculate the determinant $\det A$ of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

   ■ **Solution/comments:**

   The Rule of Sarrus gives us:

   $$\begin{vmatrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 2 \end{vmatrix}$$

   $\det A = 1 \cdot 1 \cdot 2 + 2 \cdot 0 \cdot 1 + 1 \cdot 2 \cdot 0 - 1 \cdot 1 \cdot 1 - 1 \cdot 0 \cdot 0 - 2 \cdot 2 \cdot 2 = 2 - 9 = -7$

3. Calculate the cofactor a^c_{12} of the matrix A from the previous question.
   (Note: We start counting rows and columns with 1, so a_{12} is the coefficient in the 1st row and 2nd column.)

   ■ **Solution/comments:**

   $a^c_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = -2 \cdot ((-1)^{1+1} \begin{vmatrix} 2 \end{vmatrix} + 0) = -4$

   Note: other options exist for the calculation of the cofactors of $\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$ in the recursive step.

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

Problem 5: Linear equation systems

■ **Subproblem 5.1 [2 pts]: Geometric interpretation.** Assume you are using Gaussian elimination to solve a linear equation system with three variables and three equations. You end up getting exactly one line with $0x+0y+0z=0$. What is the geometric interpretation of this outcome?

■ **Solution/comments:** The three equations represent planes in 3D that *intersect in a line*.

■ **Subproblem 5.2 [6 pts]: Gaussian elimination.** Assume the following linear equation system:

$$\begin{array}{rrrl} x & +2y & +2z & =5 \\ x & +4y & +2z & =9 \\ -x & -2y & & =1 \end{array}$$

1. Write down its augmented matrix and solve it using Gaussian elimination.

   ■ **Solution/comments:**

   The augmented matrix is:

   $$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 1 & 4 & 2 & 9 \\ -1 & -2 & 0 & 1 \end{array}\right)$$

   Multiply 1st row with -1 and $+1$ and add it to the 2nd and 3rd row, respectively:

   $$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

   Multiply 2nd row with $\frac{1}{2}$:

   $$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

   Multiply 2nd row with -2 and add it to the 1st row:

   $$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

   Multiply 3rd row with $\frac{1}{2}$:

   $$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

   Multiply 3rd row with -2 and add it to the 1st row:

   $$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

2. What is the geometric interpretation of your result?

   ■ **Solution/comments:** The three equations represent planes in 3D that *intersect in the point (-5, 2, 3)*.

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

Problem 6: Transformations

■ Subproblem 6.1 [4 pts]: Linear transformations.

1. Assume the following matrix for linear transformations in 3D:

$$A = \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   What kind of transformation do we get if we apply it to a vector in 3D?

   ■ Solution/comments: Shearing in x-direction

   Comment (not part of the answer): You can easily see this by writing down the result when applying this matrix to a generic vector (x,y,z) *which gives you* $(x+by+cz,y,z)$

2. Write down the transformation matrix for the inverse transformation of A (i.e. a matrix that transforms a vector $\vec{v}_t = A\vec{v}$ back to vector $\vec{v}$).

   ■ Solution/comments:

$$A^{-1} = \begin{pmatrix} 1 & -b & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

■ Subproblem 6.2 [10 pts]: Affine transformations.

1. Assume the following matrix for affine transformations in 3D:

$$\begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   What kind of transformation do we get if we apply it to a vector in 3D?

   ■ Solution/comments:

   Scaling of all coordinates by a factor of 3, and
   translation of all points by a vector $(2,2,2)$

   Comment (not part of the answer): You can easily see this by writing down the result when applying this matrix to a generic vector (x,y,z) *which gives you*

$$\begin{pmatrix} 3x+2 \\ 3y+2 \\ 3z+2 \end{pmatrix}$$

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

2. Complete the following sentences in a way that creates a correct statement.
   In a transformation matrix for affine transformations in 3D . . .

   - . . . the last column
     represents the image of __ the origin after applying the affine transformation
   - . . . the first 3 columns
     represent the image of __ the base vectors under the linear transformation
   - . . . the coordinates
     in the last row are called __ homogeneous coordinates

Make a mark here if you answered parts of these questions on an extra sheet: [ ]

Problem 7: Texturing

■ **Subproblem 7.1 [4 pts]: Multiple choice questions.** No explanation required.
There is only one correct answer for each individual question.

1. Which of the following statements is correct?

   The Hermite interpolation used to create Perlin noise uses ...

   A. no weights B. linear weights C. quadratic weights D. cubic weights E. sine wave weights

   ■ **Solution/comments:** The correct answer is *D. cubic weights.*

2. Which of the following statements is **not** correct?

   Bump mapping ...

   A. is used for shading calculations.
   B. uses an array of vectors.
   C. causes an actual change of the geometry.
   D. usually requires less storage than displacement mapping.
   E. usually produces worse results than displacement mapping.

   ■ **Solution/comments:** The incorrect answer (and thus the correct one to mark) is
   C. causes an actual change of the geometry..

■ **Subproblem 7.2 [2 pts]: Procedural texturing.**

Fill in the blank space in the following piece of pseudo code so that we can use it to create a stripe pattern with stripes of width π and alternating colors `color0` and `color1` along the $Z-$axis. The resulting strip pattern should have a color change at the origin.

```
stripe(x_p, y_p, z_p){

    if (______________________)
      return color0;
    else
      return color1;
}
```

■ **Solution/comments:** The solution is $\sin z_p > 0$ (or $\sin z_p \geq 0$).

■ **Subproblem 7.3 [3 pts]: Perlin noise.**

In the standard approach to calculate Perlin noise, we create random vectors (v_1, v_2, v_3) with $-1 \leq v_i \leq 1$ for $i = 1,2,3$. Yet, we only use the ones for which $(v_1^2 + v_2^2 + v_3^2) < 1$. Shortly explain why.
(Note: a short explanation is sufficient. One sentence can be enough to get full credit for this subproblem.)

■ **Solution/comments:**

By rejecting the ones with a length ≥ 1, we only pick vectors from a unit sphere, not a unit cube, thus guaranteeing that each direction has the same likelihood.

(Note: other correct explanations are possible.)

Make a mark here if you answered parts of these questions on an extra sheet: [ ]