INFOGR 2020 - Midterm Solutions

Duration: 120 minutes + 10 minutes submission period

Total points: 61

Question 1. **[1 point each = 9 points]** Answer the following understanding questions with a precise and concise explanation. You can use *annotated* sketches if necessary.

(a) Explain in your own words the difference between a scalar, a point and a vector, using an example in daily life.
(b) Describe the term unit vector.
(c) Write down and explain with an annotated sketch the Pythagoras' theorem.
(d) What is a 'orthogonal basis' for a coordinate system?
(e) For what angle between two vectors is the dot product of those vectors at its largest?
(f) What is the relation between the magnitude of a vector and the dot product of the vector with itself?
(g) Write down a general line in the slope-intercept form. What is the meaning of every term?
(h) Explain in your own words the geometric interpretation of a cross product of two vectors (in 3D).
(i) Is the dot product a vector or a scalar? And what about the cross product?

Solutions:

(a) A scalar is just a number, e.g. '3'. A point is a set of numbers, defined on a coordinate system, e.g., 1m away from both the bottom and the left edges of the table. A vector is an entity that specifies a direction and magnitude, e.g. wind velocity (speed + direction) in De uithof.

(b) A vector with length 1.

(c) Draw a triangle. Call the hypotenuse c, the base a and height b. Then $a^2 + b^2 = c^2$. In the context of vectors, the length of the vector $\vec{c}$ can be calculated using this theorem.

(d) A basis where all basis vectors are perpendicular to each other.

(e) $0°$

(f) $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$

(g) $y = mx + q$. q is the intercept, the y-coordinate of the point where the line crosses the y-axis; m is the tangent of the angle between the line and the x-axis.

(h) the cross product $\vec{w}$ of two vectors $\vec{u}$ and $\vec{v}$ is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$ and has magnitude $||\vec{w}|| = ||\vec{u}||\,||\vec{v}||\,\sin\theta$

(i) The dot product is indeed a scalar, while cross product is a vector

Question 2. **[2+5=7 points]** Consider two points $P = (3, 2)$ and $Q = (1, 4)$ in $\mathbb{R}^2$. Please answer (and *outline the intermediate steps*) for the questions below.

(a) Write down the equation of the line passing through them in implicit form.

(b) The line segment PQ is one arm of a full square $PQRS$; the vertices are labelled in the clockwise direction. Find the coordinates of R and S.

Solutions

(a) $y = -x + 5 \Rightarrow -x - y + 5 = 0$ or $y + x - 5 = 0$

Explanation: The slope of the line is $\dfrac{Q_y - P_y}{Q_x - P_x} = \dfrac{4-2}{1-3} = -1$. Equation of the straight line passing through them is therefore $y = -1x + c$. Fix $c = 5$ by the condition that the line passes through P or Q.

(b) $S = (5, 4)$ and $R = (3, 6)$.

Explanation: The length of the line segment PQ is $\sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$. The line segments PS and QR are perpendicular to PQ, and their lengths are also $\sqrt{8}$. We'll find the coordinates of R and S using from these, by shooting rays of lengths $\sqrt{8}$ from P and Q in a direction perpendicular to PQ. Given that the slope of the line segment PQ is -1, the parametric form equation of a ray shot from (x_0, y_0) in a direction perpendicular to PQ is

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \times \dfrac{1}{\sqrt{v_y^2 + (-v_x)^2}} \begin{bmatrix} v_y \\ -v_x \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \times \dfrac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. To find the location of S, we use $(x_0, y_0) = (3, 2)$ and $l = \sqrt{8}$, leading to $(3 + 2 = 5, 2 + 2 = 4)$. Similarly, to find the location of R, we use $(x_0, y_0) = (1, 4)$ and $l = \sqrt{8}$, leading to $(3, 6)$. The exercise states that PQRS are labelled in the clockwise direction, so R and S should not be swapped.

Question 3. **[4+2+3=9 points]** Given are the points in 3D: $A = (2, -1, -2)$, $B = (3, 1, -1)$ and $C = (1, -1, -1)$. Please answer (and *outline the intermediate steps*) for the questions below.

(a) Write down the general form of the implicit equation of the plane P through A, B and C.
(b) Determine the unit vectors perpendicular to P.
(c) What is the minimal distance of point $M = (5, 5, 5)$ to the plane P?

Solutions

(a) $x - y + z - 1 = 0$

The vector spanning from A to B is $u = \begin{bmatrix} 3-2 \\ 1 - -1 \\ -1 - -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and the vector spanning from A to C is $v = \begin{bmatrix} 1-2 \\ -1 - -1 \\ -1 - -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The cross product of these two vectors is $u \times v = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$.
So the equation of the plane must be of the form $x - y + z + D = 0$. The fact that all three points lies on this plane (any one will do) then leads to $D = -1$.

(b) $\pm\frac{1}{\sqrt{1^2+(-1)^2+1^2}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(c) Given Ax + By + Cz + D = 0 the distance to the point M is $d = \frac{|AM_x + BM_y + Cm_z + D|}{\sqrt{A^2+B^2+C^2}}$. This gives $\frac{|1*5+-1*5+1*5-1|}{\sqrt{1+1+1}} = \frac{4}{\sqrt{3}}$.

Question 4. **[2+7=9 points]** Given a sphere in $\mathbb{R}^3$ with centre $C = (3, 3, 3)$ and a point on the surface of the sphere $P = (2, 5, 1)$. Please answer (and *outline the intermediate steps*) for the questions below.

(a) Determine the equation for the sphere in implicit and parametric form.

(b) Determine the location of the point on the surface of the sphere closest to Q $= (6, 9, 1)$.

Solutions

(a) The implicit form equation for the sphere is given by

Your answer: $(x-3)^2 + (y-3)^2 + (z-3)^2 = 9$.

Solution: The radius of the sphere is $\sqrt{(3-2)^2 + (3-5)^2 + (3-1)^2} = 3$, so the implicit form equation for the sphere is: $(x-3)^2 + (y-3)^2 + (z-3)^2 = 9$.

parametric form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$

(b) The location of the point on the surface of the sphere closest to Q $= (6, 9, 1)$ is given by:

Your answer: $(30/7, 39/7, 15/7)$.

Solution: First note that point Q lies outside the sphere.

In order to find the answer we shoot a ray from Q towards the centre of the sphere, and let it intersect the sphere's surface. The parametric equation for this ray, with the unit vector from Q to the centre of the sphere being $\frac{1}{7}\begin{bmatrix} -3 \\ -6 \\ 2 \end{bmatrix}$, is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} + \frac{t}{7}\begin{bmatrix} -3 \\ -6 \\ 2 \end{bmatrix}$. Substituting this equation in the implicit form equation for the sphere yields the following quadratic equation for t: $t^2 - 14t + 40 = 0$, leading to the solutions $t = 4$ and $t = 10$, so the point we're looking for corresponds to $t = 4$. Use that to obtain the location of the point on the surface of the sphere closest to Q as $(30/7, 39/7, 15/7)$.

Question 5. **[2+2+2=6 points]** Given a point $P = (3, 4)$ and a circle centered around P with radius 2. Also consider two points $A = (-2, 1)$ and $B = (5, 6)$. Please answer (and *outline the intermediate steps*) for the questions below.

(a) Give the equation of the circle in implicit and parametric form.
(b) Determine the equation for line l through A and B in slope-intersect form.
(c) Write down the coordinates of one point of the intersection of l with the circle in question (a).

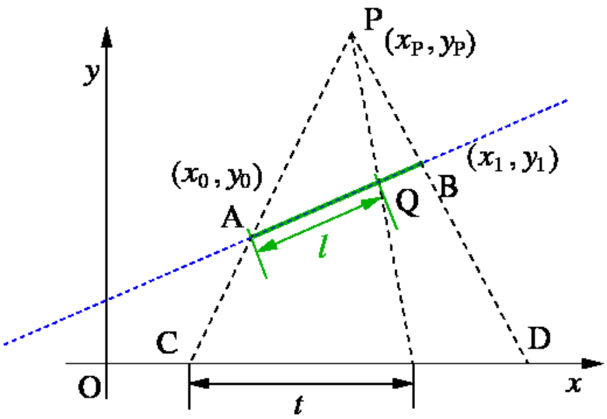
Solutions

(a) $(x-3)^2 + (y-4)^2 = 4$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

(b) $y = \frac{5}{7}x + \frac{17}{7}$; Slope $= \frac{y_2 - y_1}{x_2 - x_1}$ for b use point $x = -2, y = 1$ and solve $1 = \frac{5}{7}(-2) + b$ for $b = \frac{17}{7}$

(c) $P1 = (-2 + [\frac{25}{37} - \frac{\sqrt{70}}{37}] \cdot 7, 1 + [\frac{25}{37} - \frac{\sqrt{70}}{37}] \cdot 5) \approx (1,147, 3,248)$, $P2 = (-2 + [\frac{25}{37} + \frac{\sqrt{70}}{37}] \cdot 7, 1 + [\frac{25}{37} + \frac{\sqrt{70}}{37}] \cdot 5) \approx (4.313, 5.509)$; (1) express line as $l = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + l\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ and (2) substitute x and y into the circle equation $(x-3)^2 + (y-4)^2 = 4$; (3) resolve equation to $74l^2 - 100l + 30 = 0$ and (4) solve quadratic equation to $l_{1,2} = \frac{25}{37} \pm \frac{\sqrt{70}}{37}$, (5) insert $l_{1,2}$ into expression from (1) to get P1 and P2.

Question 6. **[2+1+4+3=10 points]** Given a set of points A, B, C, D, Q and P as shown in the figure below (at $P = (4, 5)$ there is a light source, and the shadows of A and B on the x-axis are C and D respectively). A line k passes through A, Q and B. Please answer (and *outline the intermediate steps*) for the questions below.



(a) Given that $A = (3, 2)$ and $B = (5, 3)$, give the equation for line k through A and B in the implicit and parametric form.
(b) If Q has x-coordinate 4. Determine its y-coordinate.
(c) Determine the coordinates of C and D.
(d) Determine t as a function of l (Note that Q is not fixed anymore, as in question [b]).

Solutions

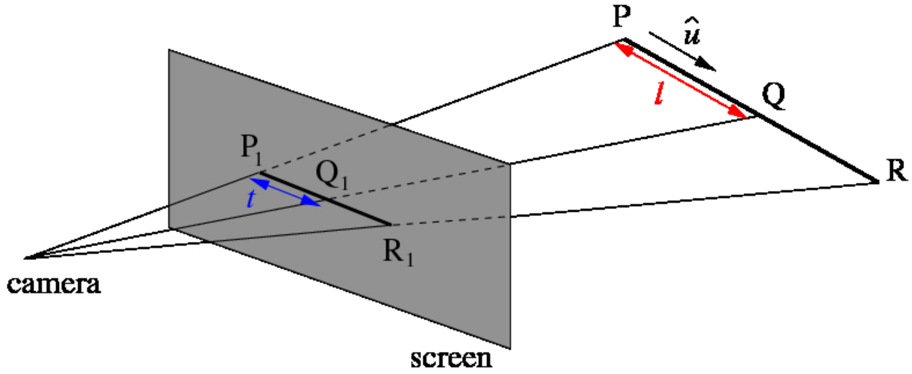
(a) $k : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{l}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for varying l. implicit form: $y = 0.5x + 0.5 \rightarrow x - 2y + 1 = 0$

(b) Determine l for this location of Q_x: $4 = 3 + \frac{2l}{\sqrt{5}}$ leading to $l = \sqrt{5}/2$ and insert for $Q_y = 2 + l * \frac{1}{\sqrt{5}} * 1 = 2 + \frac{\sqrt{5}}{2} * \frac{1}{\sqrt{5}} * 1 = 2 + 1/2 = 2.5$

(c) Shoot a ray from P to A. In parametric form, we can write: $\begin{bmatrix} x_C \\ 0 \end{bmatrix} = \begin{bmatrix} x_P \\ y_P \end{bmatrix} + \alpha \begin{bmatrix} x_A - x_P \\ y_A - y_P \end{bmatrix}$, to solve for α and then x_C. This results in $0 = 5 - 3\alpha$, so that $\alpha = 5/3$, resulting in $C = (7/3, 0)$. Doing the same for D results in $\begin{bmatrix} x_D \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \alpha \begin{bmatrix} 5 - 4 \\ 3 - 5 \end{bmatrix}$; $\alpha = \frac{5}{2}$. This results in $D = (6.5, 0)$.

(d) First, all coordinates of Q in terms of l are given from k: $Q = (3 + 2l/\sqrt{5}, 2 + l/\sqrt{5})$. Then, obtain the shadow of Q on the x-axis (call this point E). Following the procedure from question (c) gives $\begin{bmatrix} x_E \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + m \begin{bmatrix} -1 + 2l/\sqrt{5} \\ -3 + l/\sqrt{5} \end{bmatrix}$, resolve for $m = \frac{5}{3 - l/\sqrt{5}}$. Subtracting the x-coordinate of C then gives $t = m - 5/3 = \frac{5}{3 - l/\sqrt{5}} - 7/3$.

Question 7. **[6+5=11 points]** Given two points $P = (2, 3, 4)$ and $R = (5, 6, 4)$, and camera at point $E = (3, 2, -6)$. The xy-plane is the screen. Please answer (and *outline the intermediate steps*) for the questions below.



(a) Project PR to P_1R_1 on the screen as seen by the camera (see figure). Obtain the coordinates of P_1 and R_1.

(b) Given $P_1Q_1 = t$, calculate the coordinates of point Q.

Solutions

(a) Unit vector from P to E is $\hat{d}_p = (E-P)/|E-P| = \frac{1}{\sqrt{102}}\begin{bmatrix}1\\-1\\-10\end{bmatrix}$. Now solve $\begin{bmatrix}x_P\\y_P\\z_P\end{bmatrix} + k\hat{d}_p = \begin{bmatrix}x_{p1}\\y_{p1}\\0\end{bmatrix}$.

This gives:

$$\begin{cases} 2 + k\frac{1}{\sqrt{102}} & = x_{p1} \\ 3 + k\frac{-1}{\sqrt{102}} & = y_{p1} \\ 4 + k \cdot \frac{-10}{\sqrt{102}} & = 0 \end{cases} \tag{1}$$

From the third equation, we get $k = \frac{2\sqrt{102}}{5}$. The first equation gives $x_{p1} = 2 + \frac{2\sqrt{102}}{5} \cdot \frac{1}{\sqrt{102}} = 12/5$, and the second equation with the k value from above gives $y_{p1} = 3 + \frac{2\sqrt{102}}{5} \cdot \frac{-1}{\sqrt{102}} = 13/5$. So $P_1 = (\frac{12}{5}, \frac{13}{5}, 0)$.

We do the exact same thing for R and R_1, which results in a unit vector of $\hat{d}_r = -\frac{1}{2\sqrt{30}}\begin{bmatrix}-2\\-4\\-10\end{bmatrix}$.

Doing the same as above, we get:

$$\begin{cases} 5 - k \cdot \frac{-2}{2\sqrt{30}} & = x_{r1} \\ 6 - k \cdot \frac{-4}{2\sqrt{30}} & = y_{r1} \\ 4 - k \cdot \frac{-10}{2\sqrt{30}} & = 0 \end{cases} \tag{2}$$

which results in $k = \frac{4\sqrt{30}}{5}$ and $R_1 = (\frac{21}{5}, \frac{22}{5}, 0)$.

(b) $Q = (2 + \frac{5}{3\sqrt{2}}t, 3 + \frac{5}{3\sqrt{2}}t, 4)$.
There are multiple ways of calculating this. This solution will calculate Q in terms of l and Q_1 in terms of t, then determine the ratio between t and l, to get Q in terms of t.
Start with Q in terms of l. This requires a line from P to R:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix} + \frac{l}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \tag{3}$$

This gives $Q = (2 + l/\sqrt{2}, 3 + l/\sqrt{2}, 4)$. Now calculate Q_1 in terms of t, using a line from P_1 to R_1:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}12/5\\13/5\\0\end{bmatrix} + \frac{t}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \tag{4}$$

This gives $Q_1 = (\frac{12}{5} + \frac{t}{\sqrt{2}}, \frac{13}{5} + \frac{t}{\sqrt{2}}, 0)$. Drawing a line from E to Q gives (normalisation not needed):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix} + m \begin{bmatrix} -1 + l/\sqrt{2} \\ 1 + l/\sqrt{2} \\ 10 \end{bmatrix} \tag{5}$$

Filling in Q_1, because this is on that line:

$$\begin{bmatrix} \frac{12}{5} + \frac{t}{\sqrt{2}} \\ \frac{13}{5} + \frac{t}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix} + m \begin{bmatrix} -1 + l/\sqrt{2} \\ 1 + l/\sqrt{2} \\ 10 \end{bmatrix} \tag{6}$$

Using the third equation (of the z-coordinate) gives $m = 3/5$, which results in $l = \frac{5}{3}t$. That can be used to rewrite Q, to be $Q = (2 + \frac{5}{3\sqrt{2}}t, 3 + \frac{5}{3\sqrt{2}}t, 4)$.