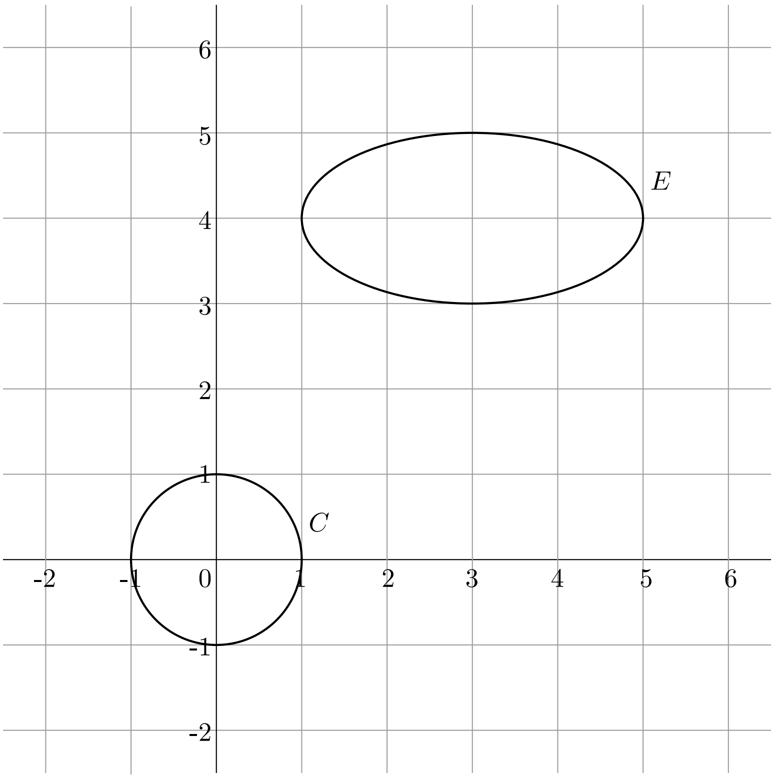
Midterm exam Graphics

Thu, Oct 5 2006, 15:15–17:15

- **Do not open this exam until instructed to do so. Read the instructions on this page first.**
- You may write your answers in English, Dutch, or German.
- You may not use books, notes, or any electronic equipment.
- Please write down your name and student ID on all your solution sheets.
- If you finish early, you may hand in your work and leave, except for the first half hour of the exam.
- If you hand in your work, write down your name and student ID on any of the lists in the front of the examination hall.
- This test consists of 2 problems with 10 subproblems in total. The maximum number of points you can score is 9.
- Your grade for the text will be $10 \cdot \min(1, \frac{p}{\pi \cdot e})$, where p is your score, π is the area of a circle with radius 1, and e is the base of the natural logarithm.

1 Circles, ellipses, and lines



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When we project a sphere onto a plane using perspective projection, we get an ellipse. In fact, an ellipse is a circle that has been scaled with different scaling factors for the two scaling directions. In the figure on the previous page, we see an ellipse with horizontal and vertical main axes. The ellipse E passes through the points $(1,4)$, $(3,3)$, $(5,4)$ and $(3,5)$.

Furthermore, we consider the line ℓ given by the following implicit equation: $\ell : 2x - 3y + 6 = 0$. This line is not shown in the figure, but it may be convenient to draw it.

The line ℓ intersects the ellipse E in two intersection points s_1 en s_2. We'd like to determine the coordinates of these intersection points. This may seem difficult, but in fact this can be done using techniques that we have seen in the Graphics lectures. It is important to realize that the ellipse in the figure is a scaled and translated version of the unit circle C (also shown in the figure).

(a) **[0 pt]** Give the parametric equation of the circle C with center $(0,0)$ and radius 1.

(b) **[1 pt]** Give the matrix M that transforms our circle C into ellipse E.

(c) **[1 pt]** Give the parametric equation of the ellipse E (use your answers to problems (a) and (b)).

Determining the intersection points of ℓ and E may seem difficult at first. But: suppose that we'd have the inverse matrix M^{-1} of M at our disposal. Then we could transform ℓ into the line $\ell' = M^{-1}\ell$. We already know that $M^{-1}E = C$. So now we can determine the intersection points s_1' and s_2' of ℓ' and C. One nice property of affine transformations is that intersections are retained. In other words: the intersections of ℓ' and C transform into the intersections of ℓ and E when we apply the matrix M. That's what we'll use,

(d) **[1 pt]** Determine the inverse M^{-1} of the matrix M that you derived in problem (b). Inverting matrices is generally a lot of work, but in this case the amount of work is only moderate. Gaussian elimination will give M^{-1} in only a few steps (but there are other means to compute M^{-1} too). Tip: how can you check whether you computed the inverse matrix correctly?

(e) **[1 pt]** We cannot apply M^{-1} to the implicit equation of ℓ; for this, we need a parametric equation. So: compute a parametric equation of ℓ.

(f) **[1 pt]** Using your solutions to problems (d) and (e), determine a parametric equation of ℓ'.

(g) **[1 pt]** Compute the intersections s_1' and s_2' of ℓ' and C. One possible approach is to equate an implicit equation of ℓ' to C. This results in a quadratic equation with two solutions.

(h) **[1 pt]** Finally, determine the intersection points s_1 and s_2 of the ellipse E and the original line ℓ.

2 Triangles, rays, and matrices

Suppose we are given an extremely simple 3D model consisting of a single triangle with vertices $(1,0,1)$, $(-2,2,0)$, and $(2,3,5)$. The viewing point of our camera is at $(7,11,15)$. We'll do ray tracing, and shoot a ray from the viewing point through the point $(5,8,11)$. We'd like to know whether this ray intersects the triangle, and if so, in what point. We can solve this in several ways. One way is to put up a system of linear equations in three unknowns, derived from the parametric equations of the ray, and of the plane in which the triangle lies.

(a) **[1 pt]** Give the system of equations, and write it down in matrix notation (i.e., in the form $Ax = b$, where A is a 3×3 matrix, and x and b are vectors). You don't have to solve the system of equations.

(b) **[1 pt]** Explain how we could use the inverse A^{-1} of the matrix A above to determine whether the ray intersects the triangle.