Final Test
Motion and Manipulation

January 3, 2007
9:00-11:00

Note: It is not allowed to consult books, notes, slides, etc. Fill out your name and student number on each page you hand in. The test consists of **six** exercises. Motivate all your answers.

1: Geometric Modeling (1.5)

(a.) Consider the tetrahedron O with vertices $p_1 = (0,0,0)$, $p_2 = (1,0,0)$, $p_3 = (0,1,0)$, and $p_4 = (0,0,1)$. Define O as an intersection of closed half-spaces $H_i = \{(x,y,z) \in \mathbb{R}^3 | f_i(x,y,z) \leqslant 0\}$.

(b.) Give an example of a non-convex semi-algebraic set O' that can be written as the intersection of a set $H_1 = \{(x,y) \in \mathbb{R}^2 | f_1(x,y) \leqslant 0\}$ and a set $H_2 = \{(x,y) \in \mathbb{R}^2 | f_2(x,y) \leqslant 0\}$. Explain your answer.

2: Configuration Space (1.5)

(a.) Determine the dimension of the configuration space for a system of three independently-moving square robots of which the first can rotate and translate, the second can only translate, and the third can only rotate.

(b.) Construct the Minkowski sum of a line segment s_1 with endpoints $(0,0)$ and $(0,1)$ and a line segment s_2 with endpoints $(1,1)$ and $(2,2)$.

(c.) Give a tight upper bound on the combinatorial complexity of the Minkowski sum of a convex polygon with n vertices and a non-convex polygon with n vertices.

3: Kinematics (2.0)

(a.) We are given a fixed orthonormal frame $F = \{f^1, f^2, f^3\}$ and a mobile orthornormal frame $M = \{m^1, m^2, m^3\}$. Initially the frames M and F coincide. We rotate M about f^1 by $\pi/3$ radians, and then translate M along f^2 by 4 units. Determine the homogeneous transformation matrix that maps mobile M coordinates into fixed F coordinates. Transform the M coordinates $(0,0,0)$ into F coordinates.

(b.) We are given a fixed orthonormal frame $F = \{f^1, f^2, f^3\}$ and a mobile orthornormal frame $M = \{m^1, m^2, m^3\}$. Initially the frames M and F coincide. We rotate M about f^1 by $\pi/6$ radians, and then translate M along m^3 by 3 units. Determine the homogeneous transformation matrix that maps mobile M coordinates into fixed F coordinates. Transform the M coordinates $(1,1,1)$ into F coordinates.

4: Combinatorial Motion Planning (2.0)

Draw the four curves

$$c_1 = \{(x,y) \,|\, x^2 + y^2 - 4 = 0\}, \;\; c_2 = \{(x,y) \,|\, x - y^2 - 4 = 0\},$$

$$c_3 = \{(x,y) \,|\, x - y = 0\}, \;\; c_4 = \{(x,y) \,|\, x + y - 6 = 0\}$$

and construct the cylindrical algebraic decomposition (or Collins' decomposition) of 2D space induced by these curves. How many two-dimensional cells do we obtain in this case?

5: Collision Detection (1.0)

Name one advantage of the use of voxel grids over kd-trees, and one advantage of the use of kd-trees over voxel grids for collision detection.

6: Manipulation (2.0)

Consider the object O given by

$$O = \{(x, y) \mid -x - 4 \leqslant 0\} \cap \{(x, y) \mid -y - 2 \leqslant 0\} \cap \{(x, y) \mid y - 2 \leqslant 0\} \cap \{(x, y) \mid x + y - 2 \leqslant 0\}.$$

(a.) Place four frictionless point contacts along the boundary of O that jointly put O in form closure. Apply Reuleaux' graphical analysis of instantaneous velocity centers to justify your answer.

(b.) Determine the three-dimensional wrench vectors corresponding to point contacts at $(-4, 0)$, $(-2, 2)$, $(-2, -2)$, and $(0, -2)$ respectively. Draw these wrenches in wrench space and determine whether or not the contacts put O in form closure. Explain your answer.