NS-202B Quantum Mechanics 2022 Midterm

You will receive a formula sheet to use with this test. A (graphical) calculator is allowed but not if it has communication capabilities. The regular time for this test is 120 minutes. This test has 11 questions on 2 pages for a total of 100 points. Motivate all your answers. Unclear and unreadable answers will be considered wrong. Use separate answer sheets for each section. Write your NAME and STUDENT NUMBER on every answer sheet. Success!

1 Wavepacket in a harmonic oscillator (Please mark your first answer sheet [1])

We consider a quantum mechanical harmonic oscillator which is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

A very useful notation is given by $\hat{H} = \hbar\omega\left(a^+a^- + \frac{1}{2}\right)$ where a^+ and a^- are the ladder operators which are defined as

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right); \qquad a^- = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right).$$

The action of the operators on the states ψ_n, with $n = 0, 1, ...$, is given by

$$a^+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad \text{and } a^-\psi_n = \sqrt{n}\psi_{n-1}.$$

Remember that $a^-\psi_0 = 0$ and $[a^-, a^+] = 1$. You can in the following use the orthonormality of the number states, i.e.,

$$\int_{-\infty}^{\infty} dx\ \psi_n^*(x)\psi_m(x) = \delta_{n,m}.$$

1. (5 points) For an energy eigenstate, we know that $\langle x\rangle$ ... (Choose 1 answer, no motivation required):

   A Oscillates with frequency ω

   B Oscillates with frequency $(n + \frac{1}{2})\omega$

   C Is constant

   D Cannot be determined due to the uncertainty principle.

2. (10 points) For the ground state, show that $\langle p^2\rangle = m\hbar\omega/2$.

3. (10 points) For the ground state, calculate $\langle x^2\rangle$ and show that ψ_0 is a minimum uncertainty state, i.e. it obeys the Heisenberg relation with an equal sign.

At time $t = 0$ we prepare the wavepacket $\psi = \frac{1}{\sqrt{2}}(\psi_0 + i\psi_1)$.

4. (5 points) Show that the state ψ is normalized.

5. (10 points) Show that the time evolution of the state is given by

$$\Psi(t, x) = \frac{e^{-i\omega t/2}}{\sqrt{2}}\left(\psi_0(x) + ie^{-i\omega t}\psi_1(x)\right).$$

6. (15 points) Calculate the time dependent expectation value of the momentum operator, i.e.,

$$\langle p(t)\rangle = \int \Psi^*(t)\hat{p}\Psi(t)dx.$$

Please turn over and use a new answer sheet

2 Finite-depth potential well (Use a new answer sheet and mark it [2])

A quantum with mass m moves along the x-axis in the finite-depth well potential

$$V(x) = \begin{cases} -V_0 & |x| < b \\ 0 & |x| \geq b \end{cases} \tag{1}$$

The depth of the well relates to its width as follows:

$$V_0 = \frac{\pi^2 \hbar^2}{16 m b^2} > 0. \tag{2}$$

It is given that the following wavefunction is normalized:

$$\psi(x) = A \begin{cases} \cos(kx) & |x| < b \\ \cos(kb) e^{-\kappa(|x|-b)} & |x| \geq b\,. \end{cases} \tag{3}$$

With the normalization constant

$$A = \sqrt{\frac{\pi}{(\pi + 4)b}}\,, \quad k = \kappa = \frac{\pi}{4b}\,. \tag{4}$$

7. (5 points) Make a sketch of the potential and of the above wavefunction and indicate the classically forbidden area.
8. (10 points) Show, using a calculation, that this wavefunction satisfies the boundary conditions, in particular at $|x| = b$ and for $|x| \to \infty$.
9. (10 points) Show that this wavefunction is an energy eigenfunction by calculating $\hat{H}\psi(x)$ in all areas, i.e. for $|x| > b$ as well as $|x| < b$.
10. (15 points) We perform a measurement of the position. Calculate the probability to find a value $|x| > b$.

    We now discuss the scattering states of this potential, i.e. states that have an energy $E > 0$.

11. (5 points) What can be said of such states? (Indicate true or false for each statement, no motivation required):

    A Scattering states are all normalized according to $\int_{-\infty}^{\infty} dx\, \psi^*(x)\psi(x) = 1$.

    B The probability current in a scattering state is always zero.

    C We can find scattering states for every positive value of the energy $E > 0$.

    D Scattering states at different energies are orthogonal.

    E Unlike in classical physics, a potential well (i.e. a potential that is nowhere positive) can cause a particle with positive energy to be scattered back.

Formula sheet for midterm exam

QM Instruction Team, Universiteit Utrecht

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Gaussian integrals

$$\int_0^\infty x^{2n} e^{-\frac{x^2}{a^2}}\,dx = \sqrt{\pi}\frac{a^{2n+1}(2n)!}{2^{2n+1}n!}$$

$$\int_0^\infty x^{2n+1} e^{-\frac{x^2}{a^2}}\,dx = \frac{n!}{2}a^{2n+2}$$

Trigonometry

$$\begin{aligned}\sin(A \pm B) &= \sin A\ \cos B \pm \cos A\ \sin B\\ \cos(A \pm B) &= \cos A\ \cos B \mp \sin A\ \sin B\end{aligned}$$

Hamiltonian

$$\hat{H} = \hat{p}^2/2m + V(x)$$

Harmonic oscillator

$$\begin{aligned}V(x) &= \frac{1}{2}m\omega^2 x^2\\ \psi_n(x) &= A_n(a^+)^n\psi_0(x)\\ E_n &= (n+\frac{1}{2})\hbar\omega\\ \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^+ + \hat{a}^-)\\ \hat{p} &= i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^+ - \hat{a}^-)\end{aligned}$$

Wavefunctions

$$\begin{aligned}\psi_n(x) &= \frac{1}{\sqrt{2^n\,n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}H_n(\xi)\\ \xi &= (m\omega/\hbar)^{1/2}x\\ H_0(x) &= 1;\ H_1(x) = 2x,\\ H_2(x) &= 4x^2-2,\\ H_3(x) &= 8x^3-12x,\\ H_4(x) &= 16x^4-48x^2+12\end{aligned}$$

Bohr radius

$$a = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$$

Fourier transforms

$$\begin{aligned}f(x) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} F(k)e^{ikx}dk\\ F(k) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\\ \phi(k) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \Psi(x,0)e^{-ikx}dx\end{aligned}$$

Time evolution of a free particle

$$\Psi(x,t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\phi(k)e^{ikx-i\underbrace{(\hbar k^2/2m)}_{\omega}t}dk$$

Phase and group velocity

$$\begin{aligned}v_\phi &= \omega/k\\ v_g &= d\omega/dk\end{aligned}$$

Jump in derivative at potential $-\alpha\delta(x)$

$$\Delta(\frac{d\psi}{dx}) = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

Probability current

$$J = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right)$$

Hamiltonian

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Wavefunctions

$$\begin{aligned} \psi_n(x) &= \frac{1}{\sqrt{2^n\, n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}H_n(\xi) \\ \xi &= (m\omega/\hbar)^{1/2}x \\ H_0(x) &= 1;\ H_1(x) = 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x, \\ H_4(x) &= 16x^4 - 48x^2 + 12 \end{aligned}$$

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