Quantum mechanics 2022 Practice Midterm Test

You will receive a formula sheet to use with this test. A (graphical) calculator is allowed but not if it has communication capabilities. The regular time for this test is 90 minutes. This test has 12 questions for a total of 100 points. Motivate all your answers. Unclear and unreadable answers will be considered wrong. Write your NAME and STUDENT NUMBER on every answer sheet. Success!

1 Uncertainty and the Delta function Potential

We consider the Dirac delta function potential well, $V(x) = -\alpha\delta(x)$, with $\alpha > 0$. This potential is known to have a single bound state with

$$E = \frac{-m\alpha^2}{2\hbar^2}.$$

1. (10 points) Show that for $x \neq 0$ the bound state wavefunction

$$\Psi(x) = Ae^{-\kappa|x|},$$

obeys the Schrödinger equation, where A is a normalization constant and $\kappa = m\alpha/\hbar^2$, where m is the mass of the particle.

2. (10 points) Show that the state is normalized when $A = \sqrt{\kappa}$.

3. (10 points) Show (using a calculation, a sketch or a symmetry consideration) that $\langle x \rangle = 0$.

4. (5 points) For a bound eigenstate $\langle p \rangle = 0$, because (Choose 1 answer, no motivation required):

   A The state is stationary so one cannot measure $p \neq 0$.

   B The state is stationary so $v = 0$.

   C The state is stationary so $\langle v \rangle = 0$.

   D The state is stationary so $\partial\Psi/\partial t = 0$.

5. (10 points) Use the Schrödinger equation to show that for any eigenstate

$$\langle p^2 \rangle = 2m(E - \langle V \rangle).$$

6. (10 points) Show that for the delta function potential

$$\langle V \rangle = -\alpha\kappa$$

7. (5 points) Show that for the delta function potential

$$\langle p^2 \rangle = \hbar^2\kappa^2, \text{ and therefore } \sigma_p = \hbar\kappa.$$

It is given that (so you do not need to calculate this)

$$\langle x^2 \rangle = \frac{1}{2\kappa^2}, \text{ and therefore, } \sigma_x = \frac{1}{\kappa\sqrt{2}}.$$

8. (10 points) Does the wavefunction obey the uncertainty principle? Motivate your answer.

9. (5 points) True or false? (Indicate true or false for each statement, no motivation required):

   A Because $|\psi|^2$ is a probability, we must have $|\psi(x)| < 1$ for all x.

   B After a precise position measurement, the momentum is zero.

   C Two measurements of the same quantity performed directly after each other without delay can have very different results.

   D A position measurement cannot increase the energy of a particle.

   E If the expectation value of the momentum is known accurately, the expectation value of the position must be uncertain.

2 Wavepacket in a harmonic oscillator

In the following we consider a quantum mechanical harmonic oscillator which is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

A very useful notation is given by $\hat{H} = \hbar\omega\left(a^+a^- + \frac{1}{2}\right)$ where a^+ and a^- are the ladder operators which are defined as

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right); \qquad a^- = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right).$$

The action of the operators on the number states $|n\rangle$, with $n = 0, 1, \ldots$ is given by

$$a^+\psi_n = \sqrt{n+1}\psi_{n+1}, \text{ and } a^-\psi_n = \sqrt{n}\psi_{n-1}.$$

Remember that $a^-\psi_0 = 0$ and $[a^-, a^+] = 1$. You can in the following use the orthonormality of the number states, i.e.,

$$\int_{-\infty}^{\infty} dx\ \psi_n^*(x)\psi_m(x) = \delta_{n,m}.$$

At time $t = 0$ we prepare the wavepacket $\psi = \frac{1}{\sqrt{2}}(\psi_0 + \psi_1)$.

10. (5 points) Show that the state ψ is normalized.

11. (10 points) Show that the time evolution of the state is given by

$$\Psi(t,x) = \frac{e^{-i\omega t/2}}{\sqrt{2}}\left(\psi_0(x) + e^{-i\omega t}\psi_1(x)\right).$$

12. (10 points) Determine the expectation value of the energy, $\langle\hat{H}\rangle$ as a function of time. Does it actually depend on time?