4 Solutions

Let op: De uitwerkingen hieronder zijn wat we gebruikt hebben voor de correctie, en soms wat beknopt. Bij het nakijken letten we er ook op dat je duidelijk maakt dat je begrijpt wat je opschrijft. Daarvoor is vaak net iets meer nodig dan wat hieronder in de uitwerkingen gegeven is.

4.1 Swinging Atwood Machine – Solution

(a) The potential energy is given by

$$U = gr\left(M - m\cos\theta\right),$$

and the kinetic energy is

$$T = \tfrac{1}{2}\left(m+M\right)\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2$$

(b) The generalized impulses are:

$$p_r = \frac{\partial T}{\partial \dot{r}} = \left(m+M\right)\dot{r} \qquad p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

(c) The Hamiltonian is:

$$\mathscr{H} = \frac{1}{2}\frac{p_r^2}{m+M} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + gr\left(M - m\cos\theta\right)$$

(d)

$$\begin{array}{rcl|rcl}
\dot{r} &=& \frac{\partial \mathscr{H}}{\partial p_r} = \frac{p_r}{M+m} & \dot{\theta} &=& \frac{\partial \mathscr{H}}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{p}_r &=& -\frac{\partial \mathscr{H}}{\partial r} = \frac{p_\theta^2}{mr^3} - g\left(M - m\cos\theta\right) & \dot{p}_\theta &=& -\frac{\partial \mathscr{H}}{\partial \theta} = -grm\sin\theta \\
\ddot{r} &=& \frac{d}{dt}\frac{\partial \mathscr{H}}{\partial p_r} = \frac{\dot{p}_r}{m+M} & \ddot{\theta} &=& \frac{d}{dt}\frac{\partial \mathscr{H}}{\partial p_\theta} = \frac{\dot{p}_\theta}{mr^2} - 2\frac{p_\theta \dot{r}}{mr^3} \\
&=& \frac{p_\theta^2}{m(m+M)r^3} - \frac{g}{m+M}\left(M - m\cos\theta\right) & &=& -\frac{g\sin\theta}{mr} - 2\frac{p_\theta p_r}{m(m+M)r^3}
\end{array}$$

(e) The EOM are non-linear coupled differential equations, $\ddot{r}$ contains terms in p_θ and $\cos\theta$; $\ddot{\theta}$ similarly has terms in p_r and r. There is no obvious way to simplify or separate the variables. (for $m/M = 3$ there is a non-obvious way, but I did not expect you to go that far)

4.2 Tumbling Blocks – Solution

(a) We can calculate the moments of inertia:

$$I = \int dx \int dy \int dz \ \rho(x,y,z) \left(\left(x^2+y^2+z^2\right) \mathbf{E}_3 - \begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix} \right)$$

The moment around the z axis I_{zz} is

$$I_{zz} = \rho \int_{-h/2}^{h/2} dz \int_0^l dy \int_0^b dx \left(x^2+y^2\right) = \tfrac{1}{3}\rho hbl \left(b^2+l^2\right) = \tfrac{1}{3}M \left(b^2+l^2\right)$$

$M = \rho V = \rho hbl$ is the total mass of the block. The products of inertia (non-diagonal terms, as discussed in class) are:

$$\begin{aligned} I_{xy} &= -\rho \int_{-h/2}^{h/2} dz \int_0^l dy \int_0^b dxxy = \tfrac{1}{4}\rho hb^2l^2 = -\tfrac{1}{4}Mbl \\ I_{xz} &= -\rho \int_{-h/2}^{h/2} dz \int_0^l dy \int_0^b dxxz = 0 \\ I_{yz} &= -\rho \int_{-h/2}^{h/2} dz \int_0^l dy \int_0^b dxyz = 0 \end{aligned}$$

(b) Torque is necessary if $\mathbf{L}$ and ω are not parallel:

$$I = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$\mathbf{L} = I \cdot \omega$, so:

$$\mathbf{L} = \begin{pmatrix} I_{xx}\omega_x + I_{xy}\omega_y \\ I_{xy}\omega_x + I_{yy}\omega_y \\ I_{zz}\omega_z \end{pmatrix}$$

Thus this is the case for rotations around the x and y axis.

(c) The smalles moments of inertia we will get for axes through the centre of mass. The three principal axis are the three axes normal to the surfaces (as in the picture).

(d) We can derive the Euler equations directly from the two formulae given:

$$\mathbf{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \qquad \dot{\mathbf{L}} + \omega \times \mathbf{L} = \mathbf{\Gamma}$$

We assume that we are in a system of principal axes, and for free rotation we set $\mathbf{\Gamma} = 0$):

$$\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2\omega_3 &= \Gamma_1 = 0 \\ \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 &= \Gamma_2 = 0 \\ \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1\omega_2 &= \Gamma_3 = 0 \end{aligned}$$

(e) We consider $\dot{\omega}_3 = 0$ and write the remaining Euler equations:

$$\begin{aligned} \dot{\omega}_1 - \left(\frac{\lambda_2 - \lambda_3}{\lambda_1}\omega_3\right)\omega_2 &= 0 \\ \dot{\omega}_2 - \left(\frac{\lambda_3 - \lambda_1}{\lambda_2}\omega_3\right)\omega_1 &= 0 \end{aligned}$$

For both we find an equation of motion of the type $\ddot{\omega}_1 + \frac{\overbrace{(\lambda_3 - \lambda_2)}^{>0}\overbrace{(\lambda_1 - \lambda_3)}^{<0}}{\lambda_1\lambda_2}\omega_3^2\omega_1 = 0$, which is a harmonic oscillation. The two oscillations are coupled (via the Euler-equation) in such a way that two oscillations are off by $\frac{\pi}{2} = 90$ degrees, i.e. if one is a cosine, the other is a sine. For the answer an explicit form of these equations should be given.

(f) For rotation around $\hat{e}_1$ we assume $\dot{\omega}_1 = 0$, and we find the same result as above; the equation of motion is $\ddot{\omega}_2 + \frac{\overbrace{(\lambda_3 - \lambda_1)}^{>0}\overbrace{(\lambda_1 - \lambda_2)}^{<0}}{\lambda_1\lambda_3}\omega_1^2\omega_2 = 0$, which is again a harmonic oscillation.

For rotation around $\hat{e}_2$, we consider $\dot{\omega}_2 = 0$ and find:

$$\begin{aligned} \dot{\omega}_1 - \left(\frac{\lambda_2 - \lambda_3}{\lambda_1}\omega_2\right)\omega_3 &= 0 \\ \dot{\omega}_3 - \left(\frac{\lambda_1 - \lambda_2}{\lambda_3}\omega_2\right)\omega_1 &= 0 \end{aligned}$$

For both we find an equation of motion $\ddot{\omega}_1 + \frac{\overbrace{(\lambda_2 - \lambda_3)}^{<0}\overbrace{(\lambda_1 - \lambda_2)}^{<0}}{\lambda_1\lambda_2}\omega_3^2\omega_1 = 0$, which is an exponentially growing solution.

note: if students follow the hints, they will find solutions with $e^{\pm iCt}$ for $\hat{e}_1$ and $\hat{e}_3$ (harmonic), and solutions with $e^{\pm Ct}$ for $\hat{e}_2$ (exponential).

(g) Rotations around the principal axis with the largest and the principal axis with the smallest moment are *stable*, as small disturbances result in limited, harmonic motion around the main rotational axis. Rotations around the principal axis with the non-extreme moments are *unstable*, as small disturbances grow exponentially with time.

4.3 Lagrange Points – solution

a) Specify all forces (including fictitious ones) on m and their direction:

There are two gravitational forces, F_{13}, and F_{23}, pointing from m to the respective masses (M_1 and M_2). The centrifugal force points raidally rouward from the center of mass. The Coriolis force points in the direction $v \times \Omega$, e.g. if $\mathbf{v}$ points in the positive x direction, the Coriolis force would point outwards, along y (assuming Ω is positive).

b) $v = 0$, so the only forces are two gravitational forces and the centrifugal force.

The centrifugal force is

$$F_{cf} = m\Omega^2 r\hat{\mathbf{r}} = m\Omega^2(x, y)$$

The position of M_1 in the center of mass is $(-M_2/(M_1 + M_2)R, 0) = (-\alpha R, 0)$, so that

$$F_{13} = \frac{GM_1 m}{((x + \alpha R)^2 + y^2)^{\frac{3}{2}}}(-(x + \alpha R), -y)$$

and the position of M_2 in the center of mass is $(M_1/(M_1 + M_2)R, 0) = (\beta R, 0)$, so that

$$F_{23} = \frac{GM_2 m}{((x - \beta R)^2 + y^2)^{\frac{3}{2}}}(-(x - \beta R), -y)$$

The total force is:

$$F_{tot,x} = m\Omega^2 x + \frac{-GM_1 m(x + \alpha R)}{((x + \alpha R)^2 + y^2)^{\frac{3}{2}}} + \frac{-GM_2 m(x - \beta R)}{((x - \beta R)^2 + y^2)^{\frac{3}{2}}}$$

$$F_{tot,y} = m\Omega^2 y + \frac{-GM_1 my}{((x + \alpha R)^2 + y^2)^{\frac{3}{2}}} + \frac{-GM_2 my}{((x - \beta R)^2 + y^2)^{\frac{3}{2}}}$$

This can be further simplified by noticing:

$$GM_2 = G\alpha(M_1 + M_2) = \alpha\Omega^2 R^3$$

(which is needed in the next question)

c)

$$F_\perp = F_{tot} \cdot \frac{1}{\sqrt{x^2 + y^2}}(y, -x) = \frac{1}{\sqrt{x^2 + y^2}}(F_{tot,x}y - F_{tot,y}x)$$

Note that the centrifugal force has no transverse component. Using also immediately Eq b

$$\begin{aligned} F_\perp &= \frac{m\Omega^2 R^3}{\sqrt{x^2 + y^2}}\left(\frac{-\beta\alpha R y}{((x + \alpha R)^2 + y^2)^{\frac{3}{2}}} + \frac{\alpha\beta R y}{((x - \beta R)^2 + y^2)^{\frac{3}{2}}}\right) \\ &= \frac{m\Omega^2 R^4 \alpha\beta y}{\sqrt{x^2 + y^2}}\left(\frac{1}{((x - \beta R)^2 + y^2)^{\frac{3}{2}}} - \frac{1}{((x + \alpha R)^2 + y^2)^{\frac{3}{2}}}\right), \end{aligned}$$

where the terms with xy in the numerator immediately canceled.

This component is zero when $(x + \alpha R)^2 = (x - \beta R)^2$, meaning that the x-coordinate of the Langrage points is in the middel between the two masses at $x_{M_1} = -\alpha R$ and $x_{M_2} = \beta R$.

d)
$$F_{\|} = F_{tot} \cdot \frac{1}{\sqrt{x^2+y^2}}(x,y) = \frac{1}{\sqrt{x^2+y^2}}\left(F_{tot,x}x - F_{tot,y}y\right)$$

Note that the centrifugal force is $F_{\|,cf} = m\Omega^2\sqrt{x^2+y^2}$

Using the fact that $(x_L+\alpha R) = (x_L-\beta R)$ for the Lagrange points, we can write the denominators of the gravitational forces all as $((x_L+\alpha R)^2+y^2)^{2/3}$ (or $((x_L-\beta R)^2+y^2)^{3/2}$), so that

$$\begin{aligned} F_{\|} &= \frac{m\Omega^2R^3}{\sqrt{x^2+y^2}}\left(\frac{x^2+y^2}{R^3} + \frac{-\alpha(x+\beta R)x - \beta(x-\alpha R)x - \alpha y^2 - \beta y^2}{((x_L+\alpha R)^2+y^2)^{2/3}}\right) \\ &= \frac{m\Omega^2R^3}{\sqrt{x^2+y^2}}\left(\frac{x^2+y^2}{R^3} + \frac{-(\alpha+\beta)x^2 - (\alpha+\beta)y^2}{((x_L+\alpha R)^2+y^2)^{3/2}}\right) \\ &= m\Omega^2R^3\sqrt{x^2+y^2}\left(\frac{1}{R^3} - \frac{1}{((x_L+\alpha R)^2+y^2)^{3/2}}\right) \end{aligned}$$

which is zero if the distance between the Lagrange point and M_1 ($\sqrt{(x_L+\alpha R)^2+y^2}$) is equal to R, i.e. the distance between M_1 and M_2, so L_4 and L_5 lie on the apex of the two equilateral triangles with M_1, M_2 at the corners.

e) The restoring force is provided by the Coriolis force. As discussed in class, there are stable orbits around the Lagrange points.