Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Retake Exam, March 13, 2014

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Tip: Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Kepler problem

Consider a particle of reduced mass μ orbiting in a central force with potential energy $U = kr^n$ with $kn > 0$. **[total: 35 points]**

(a) Explain what the condition $kn > 0$ tells us about the force. (3 points)

(b) With given angular momentum ℓ sketch the effective potential energy U_{eff} for the cases $n = 2, -1$ and -3. (3+3+3 points)

(c) Find the radius at which the particle (with given angular momentum ℓ) can orbit at a fixed radius. For which value of n is this circular orbit stable (do your sketches confirm this conclusion)? (6+4 points)

(d) For the stable case show that the period of small oscillations about the circular orbit is $\tau_{osc} = \tau_{orb}/\sqrt{n+2}$. (6 points)

(e) Argue that if $\sqrt{n+2}$ is a rational number then these orbits are closed. (7 points)

2 Beads on a Bike's Spokes

We are considering a slightly idealized motion of a bead of mass m along the spoke of a rotating bicycle wheel: the bead can move only radially, and the moment of inertia of the wheel is given by $\mathbf{I} = MR^2$ (with M the mass of the wheel, and R its radius). The wheel is off the ground and rotating freely around its axis, so the only force you need to consider is gravity. **[total: 30 points]**

Figure 1: Bicycle Spoke Beads.

(a) Show that the Lagrangian of this system is given by

$$\mathcal{L}(\phi, \dot{\phi}, r, \dot{r}, t) = \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - mgr\cos\phi,$$

where the angle ϕ describes the rotation of the wheel and r is the distance of the bead from the axis. (8 points)

(b) Set up Lagrange Equations for this system, find the generalized forces and generalized momenta, and write out the equations of motion for r and ϕ. (8 points)

(c) The equations of motion are coupled and not easy to solve. If we had used the Hamiltonian $\mathcal{H}$ instead of the Lagrangian $\mathcal{L}$ to find the equations of motion, would they (1) look different and (2) be easier to solve? Give an argument for your answers! (4 points)

(d) It is a reasonabe assumption that the mass of the wheel M is much larger than the mass of the bead m, and that in order to solve the differential equation in r we can thus consider $\dot{\phi} = \omega = \text{const}$. Find the solution for $r(t)$ for the initial values $r(0) = \frac{1}{2}R$ and $\dot{r}(0) = 0$. If you cannot solve the differential equation, partial credit will be given if you can correctly identify the general form of the solution. (6 points)

(e) The solution for $r(t)$ is not periodic and will grow to infinity. In reality, beads on bike wheels do not usually exhibit this behaviour. Argue from the forces acting on the bead and their change in time, that – in hindsight – we should have expected a non-periodic motion of the bead. Also, which physical limitations or properties of the system are not part of the above Lagrangian, leading to the unphysical answer? (4 points)

3 Tumbling Rotations

In this question, we will deal with the rotation of rigid bodies in three dimensions. Take a rectangular box of mass M and homogenous density ρ. As shown in the figure on the right, the box has dimensions $h < b < l$ and is rotating around a point along one of its shortest edges. **[total: 35 points]**

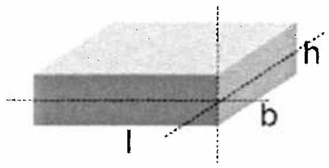

Figure 2: Rectangular Prism with homogeneous mass distribution. The origin is in the centre of one of the four shortest edges.

(a) Calculate the inertial tensor $\mathbf{I}$ around the given axes and origin. (6 points)

(b) The inertial tensor consists of *moments of inertia* and *products of inertia*. Explain these two terms and give the physical meaning of the two elements of the inertial tensor I_{zy} and I_{yz} for a rotation $\vec{\omega}$ around an arbitrary axis. Argue from symmetry whether one of those two elements has to always be larger than the other. (3 points)

(c) Around which of the three axes given in the figure can the body only rotate if there is an external torque? Give an explanation why this is so. (3 points)

(d) Are any of the axes given in the figure *principal axes* of the body? Start by explaining what a "prinicpal axis" is! (2 points)

Let us now consider rotations of the body occuring without an external torque (free rotations):

(e) Through which point (origin) do we find the principal axes which correspond to the smallest *principal moments* λ_i? What is a "principal moment" and which direction will the principal axes have in this case? (3 points)

(f) Find the general Euler Equations (no external torque) for the box. How many different *principal moments* do we find? (3+1 points)

In the final exam this year, we have considered Earth to be a spinning top with $\lambda_1 = \lambda_2 \neq \lambda_3$. This led us to find a harmonic disturbance of the main rotational axis $\hat{e}_3$, namely the "Chandler Wobble". Some of you pointed out that the Earth actually does have *three* distinct principal moments $\lambda_1 < \lambda_2 < \lambda_3$. Let us find out what this means for the stability of harmonic disturbances:

(g) We start with a rotation mainly around $\hat{e}_3$, or in other words $\omega_3 \gg \omega_2, \omega_1$. Using the Euler Equations, show that this means that ω_3 is close to constant. (2 points)

(h) Use the above fact to simplify the Euler Equations for ω_1 and ω_2 and solve the differential equations. (4 points)
(Hint: a useful trial solution is $\omega_{1,2}(t) = Ae^{\kappa t}$; also, remember that $\lambda_1 < \lambda_2 < \lambda_3$).

(i) Now repeat the calculations in (h) for the case that the initial rotation was (1) mainly around $\hat{e}_1$ and (2) mainly around $\hat{e}_2$. (3+3 points)

(j) While the solutions you found in (h) and (i) look similar, there are important differences in the long-term behaviour. Discuss these differences in terms of stabilty (or lack of stability) of the initial rotation. Is the "Chandler Wobble" stable? (2 points)

Handige formules

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\arcsin(\gamma)) = \sqrt{1-\gamma^2} \qquad \cos(\pi + \alpha) = -\cos\alpha \qquad \sin(\pi + \beta) = -\sin\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right) \qquad \sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}}$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}\left(\epsilon^2 - 1\right) \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad \frac{c_1}{1+\epsilon_1\cos(\phi+\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi+\delta_2)}$$

$$U_{\mathrm{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\mathrm{rel}} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad J_{ij} = I_{ij}^{\mathrm{cm}} + m\left(||\vec{a}||^2\delta_{ij} - a_i a_j\right)$$

$$\mathbf{I} = \iiint dV\ \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

$$\left(\frac{dQ}{dt}\right)_{S_o} = \dot{Q} + \Omega \times Q \qquad \left(\frac{d^2Q}{dt^2}\right)_{S_o} = \ddot{Q} + 2\Omega\times\dot{Q} + \Omega\times(\Omega\times Q)$$

$$\mathcal{L} = T - V \qquad \frac{\partial\mathcal{L}\left(q(t),\dot{q}(t),t\right)}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}\left(q(t),\dot{q}(t),t\right)}{\partial\dot{q}} = 0$$

$$\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = T + V \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \quad \dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \qquad \dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma}$$

$$\lambda\vec{\omega} = \mathbf{I}\vec{\omega} \qquad (\mathbf{I} - \lambda\mathbf{E_3})\vec{\omega} = 0 \qquad \det(\mathbf{I} - \lambda\mathbf{E_3}) = 0 \qquad \mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\int x^2\sqrt{R^2 - x^2}\,dx = \frac{1}{8}\left(x\sqrt{R^2-x^2}\left(2x^2 - R^2\right) + R^4\arctan\frac{x}{\sqrt{R^2-x^2}}\right)\Bigg|_0^R$$

$$\arctan 0 = 0 \qquad \arctan\pm\infty = \pm\tfrac{1}{2}\pi$$

$$f(x^*) = x^*, \qquad f(x_a) = x_b \wedge f(x_b) = x_a \Rightarrow f(f(x^*)) = x^*$$