Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Final Exam, January 29, 2015

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
To make life easier for our TAs, *if possible* please answer the questions in English. Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Coupled Oscillations — bead on a hoop

Consider the system in which a bead of mass m is threaded on a frictionless wire hoop of radius R and mass m (i.e., the bead can slide frictionlessly along the hoop). The hoop is suspended at point A and is free to swing in its own vertical plane, as shown in Fig. 1. **[total: 17 points]**

a) Using the variables ϕ_1 and ϕ_2 as shown in Fig. 1, where ϕ_1 is the angle subtended by the center C of the hoop to the vertical line through A, and ϕ_2 is the angle subtended by the particle to the vertical line through C, write down the Lagrangian of the system. (5 points)

b) Obtain the equations of motion for ϕ_1 and ϕ_2. (2 points)

c) From the equations of motion in part b, identify the normal modes (4 points) and normal frequencies (4 points) for small oscillations.

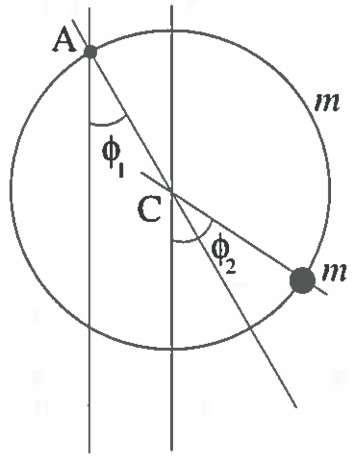

Figure 1: Bead on a hoop

d) Using the normal modes and frequencies, write down the full solutions for $\phi_1(t)$ and $\phi_2(t)$. (2 points)

2 Non-Point-Masses: Rotation and Deformation

In this question, we will take a step away from point sources and deal with three dimensional bodies. Take a rectangular box of mass M and homogeneous density ρ. As shown in the figure on the right, the box has dimensions $h < b < l$ and is rotating around a point in the middle of one of its shortest edges. We will start by assuming that the body is *rigid*, *i.e.* the relative distance of all parts (and thus the shape) of the body cannot change. **[total: 22 points]**

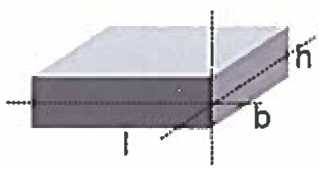

Figure 2: Rectangular Prism with homogeneous mass distribution.

(a) Calculate the inertial tensor **I** around the given axes and the origin in the middle of the shortest edge of the box. (4 points)

(b) The inertial tensor consists of *moments of inertia* and *products of inertia*. Explain these two terms and give the physical meaning of the two elements of the inertial tensor I_{zy} and I_{yz} for a rotation $\vec{\omega}$ around an arbitrary axis. Argue from symmetry whether one of those two elements has to always be larger than the other. (3 points)

(c) Around which of the three axes given in the figure can the body only rotate if there is an external torque? Give an explanation why this is so. (3 points)

(d) Are any of the axes given in the figure *principal axes* of the body? Start by explaining what a "principal axis" is! (2 points)

(e) Through which point (origin) do we find the principal axes which correspond to the smallest *principal moments* λ_i? What is a "principal moment" and which direction will the principal axes have in this case? (3 points)

In reality, no body is not completely rigid. We have found a generalized Hooke's law describing the relationship between the surface forces ("stress" tensor) and deformation in the form of the relevant parts of the derivatives matrix ("strain" tensor):

$$\mathbf{\Sigma} = \alpha e \mathbf{1} + \beta \mathbf{E}'.$$

For the next section, $\mathbf{\Sigma} = \begin{bmatrix} xz & z^2 & 0 \\ z^2 & 0 & -y \\ 0 & -y & 0 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(f) Find the surface force on a small area dA of the surface $x^2 + y^2 + 2z^2 = 4$ at the point $(1, 1, 1)$. Which type(s) of surface forces do you find? (3+1 point)

(g) Which two types of deformation does a strain tensor contain, and where can you find them in the tensor? Which type of strain is given by the given tensor $\mathbf{E}$? (2+1 point)

3 Hamiltonian — bead sliding down a cylinder

A uniform cylinder of radius a, density ρ and total mass M is mounted so as to rotate freely around a vertical axis. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track along which a mass point m can slide without friction (see Fig. 3). Suppose a particle starts at rest at the top of the cylinder and slides down under the influence of gravity. Uniform spiral means that while following the spiral downwards, *with respect to the cylinder*, if the particle goes around its axis by an angle θ, then vertically it drops by a distance $z = \alpha\theta$, with α being a constant.

Use θ for the rotation of the particle *with respect to the cylinder* around its axis, and ϕ for the rotation of the cylinder around its axis. **[total: 18 points]**

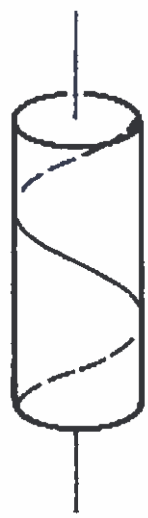

Figure 3: Bead moving helically around a cylinder.

a) Write down the Lagrangian of the system. (4 points)

b) Show that, using the Lagrangian in a, and

$$\gamma = ma^2 \left[\left(1 + \frac{\alpha^2}{a^2}\right)\left(1 + \frac{M}{2m}\right) - 1 \right],$$

that the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2\gamma}\left[\left(1+\frac{M}{2m}\right)p_\theta^2 - 2p_\theta p_\phi + \left(1+\frac{\alpha^2}{a^2}\right)p_\phi^2\right] - mg\alpha\theta.$$

(4 points)

c) Write down the equations of motion for θ, ϕ, p_θ and p_ϕ (1+1+1+1 point). Identify the conserved quantity of motion from these equations of motion. (1 point) What does this conserved quantity physically mean? (1 point)

d) Using the conserved quantity as a constant c, derive the behaviour of $\theta(t)$ and $\phi(t)$ as a function of time. (2+2 points)

4 Mapping Chaos, Logistically

In class we have used the driven damped pendulum to study chaotic motion. A somewhat simpler system that also displays chaotic behaviour is the iterative Logistic Map

$$x_{t+1} = f(x_t) = r \cdot x_t\,(1 - x_t),$$

where x_t is a 'population' at time t, and r $(0 < r < 4)$ is a constant. The Logistics map for different values of r, and the second iterative map for $r = 3.5$ are shown in figure 4. In the figure, you see several fixed points of the Logistic map. $x^* = 0$ and $x^* = (r-1)/r$ are always fixed points. **[total: 13 points]**

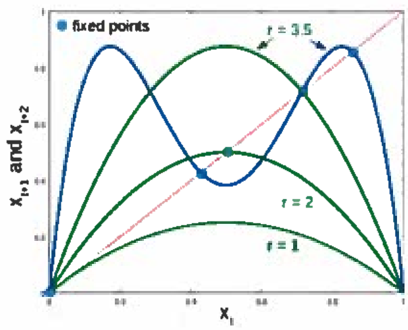

Figure 4: Logistic Map for different values of r. At $r = 3.5$, both first and second iterative map $f(f(x))$ are shown.

a) Some general questions about Chaotic Systems:

(1) What type of non-linear function are we dealing with in the case of the Logistic Map? Do all non-linear functions exhibit chaotic behaviour? (1+1 points)

(2) r is the "control parameter" of the Logistics Map. Give a short definition of this term. (1 point)

b) Verify that fixed-point equation for the second-iterate $f(f(x))$ of the Logistics Map has the form $rx(x - \frac{r-1}{r})(r^2x^2 - r(r+1)x + r + 1) = 0$, and calculate the four fixed points. (2+2 points)

c) The two-cycle shown in the figure for $r = 3.5$ is not stable. Give a definition of stability of a fixed point, and show that a stable two-cycle exists for $3 < r < 1 + \sqrt{6}$. (1+2 points)

d) r_n denotes the value of the control parameter, where a 2^n-cycle becomes unstable. We know $r_0 = 3.0$, and you have proven $r_1 = 3.4495$. You will encounter r_2, r_3, and r_4 at 3.5441, 3.5644, and 3.5688, respectively. Which phenomenon of a chaotic system is evident here, and which constant can be used to describe it? (name + definition!) (1+1+1 points)

Useful Formulas

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\arcsin(\gamma)) = \sqrt{1-\gamma^2} \qquad \cos(\pi+\alpha) = -\cos\alpha \qquad \sin(\pi+\beta) = -\sin\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)) \qquad \sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}}$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1) \quad \mu = \frac{m_1 m_2}{m_1+m_2} \quad \frac{c_1}{1+\epsilon_1\cos(\phi+\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi+\delta_2)}$$

$$U_{\text{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\text{rel}} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad J_{ij} = I_{ij}^{\text{cm}} + m\left(||\vec{a}||^2\delta_{ij} - a_i a_j\right)$$

$$\mathbf{I} = \iiint dV\ \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

$$\text{sphere: } I_{zz} = \frac{2}{5}mr^2 \qquad \text{cylinder: } I_{zz} = \frac{1}{2}mr^2 \qquad \text{hoop: } I_{zz} = mr^2$$

$$\left(\frac{dQ}{dt}\right)_{S_o} = \dot{Q} + \Omega\times Q \qquad \left(\frac{d^2Q}{dt^2}\right)_{S_o} = \ddot{Q} + 2\Omega\times\dot{Q} + \Omega\times(\Omega\times Q)$$

$$\mathcal{L} = T - V \qquad \frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial\dot{q}} = 0$$

$$\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = T + V \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \quad \dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \qquad \dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma}$$

$$\lambda\vec{\omega} = \mathbf{I}\vec{\omega} \qquad (\mathbf{I}-\lambda\mathbf{E_3})\vec{\omega} = 0 \qquad \det(\mathbf{I}-\lambda\mathbf{E_3}) = 0 \qquad \mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_i(d\vec{A}) = \sum_j \sigma_{ij} dA_j \qquad \vec{F}(d\vec{A}) = \Sigma d\vec{A} = \Sigma\hat{n}dA$$

For a surface and a point on that surface: $S(x,y,z) = 0,\ P \in S: \quad \vec{n} = \nabla S$

$\int_0^{2\pi}\sin^n(x)\cos^m(x)dx = 0$ for odd m or n. $\qquad \int_0^{2\pi}\cos^2(x)dx = \int_0^{2\pi}\sin^2(x)dx = \pi$

$$\arctan 0 = 0 \qquad \arctan\pm\infty = \pm\tfrac{1}{2}\pi$$

$$f(x^*) = x^* \qquad f(x_a) = x_b \wedge f(x_b) = x_a \Rightarrow f(f(x^*)) = x^* \qquad x_t = x^* + \epsilon \Rightarrow \epsilon_{t+1} \approx f'(x^*)\epsilon_t$$