Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Final Exam, January 28, 2016

Mark on *each* sheet clearly your name and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
To make life easier for our TAs, *if possible* please answer the questions in English. Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Coupled Oscillations

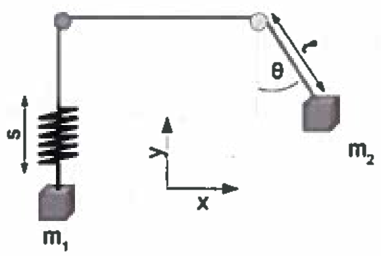
As a model of a linear triatomic molecule (such as CO_2), consider the system shown in Fig. 1, with two identical atoms each of mass m connected by two identical springs to a single atom of mass M. To simplify matters, assume that the system is confined to move in one dimension. [total 15 pt]

(a) Write down the Lagrangian and find the normal frequencies of the system. (2+8 points)

(b) One of the eigenvalues of the characteristic matrix is zero. Once you use the zero eigenvalue in the characteristic matrix, you will find that the following symmetry property of it; namely that the sum of all the row elements (and also the column elements) is zero. Use this property to "guess" the eigenvector corresponding to the zero eigenvalue. Provide the physical reason (look at the figure!) for why this eigenvector must lead to a zero eigenvalue. (2+3 points)

2 Machine of Atwood with a spring

Two masses m_1 en m_2 are connected by a mass-less line of length L which runs without friction over two pulleys . Between mass 1 and the line there is an ideal spring with a spring constant k; the motion of mass 1 is limited to the vertical y-direction. Mass 2, on the other hand, can move freely in the plane of the drawing. The extension s of the spring is measured from its equilibrium point, not from the point of zero force on the spring.



You may assume that the motions occur in such a manner that neither spring nor masses ever jump over the pulleys. Please note that part *e)* is on the next page. [total: 17 pt]

a) Calculate the potential energy U and the kinetic energy T of this system in the given coordinates. (5 point)

b) Calculate the Lagrangian $\mathcal{L}$ and the generalized moments p_ℓ, p_s en p_θ. (4 point)

c) From this, find the Hamiltonian $\mathcal{H}(\ell, s, \theta, p_\ell, p_s, p_\theta)$ and show that it is equal to $T+U$. (5 point)

d) Which conserved quantity in this system is *not* connected to an ignorable coordinate? (1 point)

e) The Hamilton Formalism ($\mathcal{L}$, $\mathcal{H}$) and the laws of Newton are equivalent, yet some systems are easy to solve in one of them, while difficult in the other. Give one example each – including a short(!) argument for your choices – where (i) it is better to use the Hamilton formalism and (ii) where directly using the laws of Newton is the better choice. (2 points)

3 One dimensional map

Consider the one-dimensional map

$$x_{n+1} = \begin{cases} 3x_n & \text{for } 0 \leq x_n \leq 1/3 \\ \frac{3}{2}(1 - x_n) & \text{for } 1/3 < x_n \leq 1 \end{cases}$$

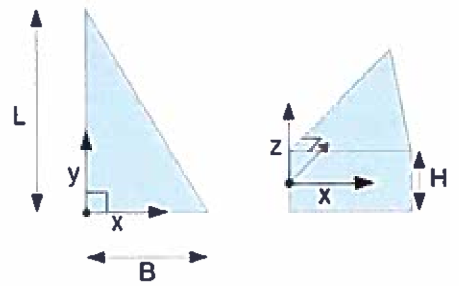
[total: 13 pt]

(a) Find the fixed points of this map and analyze their stability. (2+2 points)

(b) Sketch the second iterate, x_{n+2}, of the map. Find the location of the period two orbits, and analyze their stability. (5+4 points)

4 Rigid Body Rotations and Chandler Wobble

We are considering rotational motion of a rigid body with homogeneous mass distribution (mass M, density ρ). Let us start with a triangular prism where $H < B < L$. [total: 15 pt]



(a) Calculate the inertial tensor $\mathbf{I}$ of the prism relative to the centre of rotation and the axes shown in the figure! (4 points)

(b) One of the given axes is a principal axis ("hoofdas"). Identify this principal axis from the found inertial tensor and symmetry considerations, and in doing so define the term "principal axis". Why do you find a principle axis for this body though it does not possess rotational symmetry? (4 points)

For the second part of this problem we consider a rigid body with only two different principal moments $\lambda_1 = \lambda_2 \neq \lambda_3$. Our *body frame*, as per definition, is fixed in the centre of mass and its axis are pointing along the principal axis of the body.

(c) Write out the three Euler Equations for this rigid body without external torque. (2 points)

(d) The rotation of Earth is well described by the Euler Equations that you have found ($\hat{e}_3$ is the rotational axis of earth, $\lambda_3 \approx 306/305\lambda_1$). The Euler Equations suggest that the direction of the angular velocity $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ of Earth changes in time. Find the period of this "Chandler Wobble". (3 points)

(e) In reality, the Chandler Wobble has a period of 433 days, which is not the period you have found. Give two potential explanations that could explain this discrepancy – other than that we used the wrong pricipal axes or principal moments. (2 points)

Useful Formulas

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\arcsin(\gamma)) = \sqrt{1-\gamma^2} \qquad \cos(\pi+\alpha) = -\cos\alpha \qquad \sin(\pi+\beta) = -\sin\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)) \qquad \sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}}$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1) \quad \mu = \frac{m_1 m_2}{m_1+m_2} \quad \frac{c_1}{1+\epsilon_1\cos(\phi+\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi+\delta_2)}$$

$$U_{\text{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\text{rel}} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad J_{ij} = I_{ij}^{\text{cm}} + m\left(\|\vec{a}\|^2\delta_{ij} - a_i a_j\right)$$

$$\mathbf{I} = \iiint dV\ \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

$$\text{sphere: } I_{zz} = \frac{2}{5}mr^2 \qquad \text{cylinder: } I_{zz} = \frac{1}{2}mr^2 \qquad \text{hoop: } I_{zz} = mr^2$$

$$\left(\frac{dQ}{dt}\right)_{S_o} = \dot{Q} + \Omega \times Q \qquad \left(\frac{d^2Q}{dt^2}\right)_{S_o} = \ddot{Q} + 2\Omega \times \dot{Q} + \Omega \times (\Omega \times Q)$$

$$\mathcal{L} = T - V \qquad \frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial\dot{q}} = 0$$

$$\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = T + U \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \quad \dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \qquad \dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}$$

$$\lambda\vec{\omega} = \mathbf{I}\vec{\omega} \qquad (\mathbf{I} - \lambda\mathbf{E_3})\vec{\omega} = 0 \qquad \det(\mathbf{I} - \lambda\mathbf{E_3}) = 0 \qquad \mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_i(d\vec{A}) = \sum_j \sigma_{ij} dA_j \qquad \vec{F}(d\vec{A}) = \Sigma d\vec{A} = \Sigma\hat{n}\,dA$$

For a surface and a point on that surface: $S(x,y,z) = 0,\ P \in S:\quad \vec{n} = \nabla S$

$\int_0^{2\pi}\sin^n(x)\cos^m(x)dx = 0$ for odd m or n. $\qquad \int_0^{2\pi}\cos^2(x)dx = \int_0^{2\pi}\sin^2(x)dx = \pi$

$$\arctan 0 = 0 \qquad \arctan\pm\infty = \pm\tfrac{1}{2}\pi \qquad \det(\mathbf{K} - \omega^2\mathbf{M}) = 0$$

$$f(x^*) = x^* \qquad f(x_a) = x_b \wedge f(x_b) = x_a \Rightarrow f(f(x^*)) = x^* \qquad x_t = x^* + \epsilon \Rightarrow \epsilon_{t+1} \approx f'(x^*)\epsilon_t$$