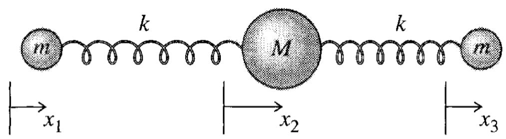
Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Final Exam, January 28, 2016

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
To make life easier for our TAs, *if possible* please answer the questions in English. Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Coupled Oscillations

As a model of a linear triatomic molecule (such as CO_2), consider the system shown in Fig. 1, with two identical atoms each of mass m connected by two identical springs to a single atom of mass M. To simplify matters, assume that the system is confined to move in one dimension. [total 15 pt]



(a) Write down the Lagrangian and find the normal frequencies of the system. (2+8 points)

(b) One of the eigenvalues of the characteristic matrix is zero. Once you use the zero eigenvalue in the characteristic matrix, you will find that the following symmetry property of it; namely that the sum of all the row elements (and also the column elements) is zero. Use this property to "guess" the eigenvector corresponding to the zero eigenvalue. Provide the physical reason (look at the figure!) for why this eigenvector must lead to a zero eigenvalue. (2+3 points)

1.1 Solution

See figure 1 for the solution.

We will introduce the shorthand notation $\lambda = M/m$.

(a) The total KE is

$$\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 = \frac{1}{2}m(\dot{x}_1^2 + \lambda\dot{x}_2^2 + \dot{x}_3^2).$$

The total PE is

$$\frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 = \frac{1}{2}k(x_1^2 + 2x_2^2 + x_3^3 - 2x_1x_2 - 2x_2x_3).$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + \lambda\dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}k(x_1^2 + 2x_2^2 + x_3^3 - 2x_1x_2 - 2x_2x_3).$$

(b) The Lagrange equations read

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x_1} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_1} &\implies&\quad m\ddot{x}_1 = -kx_1 + kx_2 \\
\frac{\partial\mathcal{L}}{\partial x_2} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_2} &\implies&\quad m\lambda\ddot{x}_2 = kx_1 - 2kx_2 + kx_3 \\
\frac{\partial\mathcal{L}}{\partial x_3} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_3} &\implies&\quad m\ddot{x}_3 = kx_2 - kx_3.
\end{aligned}$$

This can be expressed as the matrix equation $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$, where

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & \lambda m & 0 \\ 0 & 0 & m \end{bmatrix}, \text{ and } \mathbf{K} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}.$$

(c) To find the normal frequencies, we must set $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$, where

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - \lambda m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{bmatrix}.$$

This determinant is a bit easier to deal with if we divide through by m and recall that $k/m = \omega_0^2$:

$$\frac{1}{m}\left(\mathbf{K} - \omega^2\mathbf{M}\right) = \begin{bmatrix} \omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \lambda\omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0^2 - \omega^2 \end{bmatrix}.$$

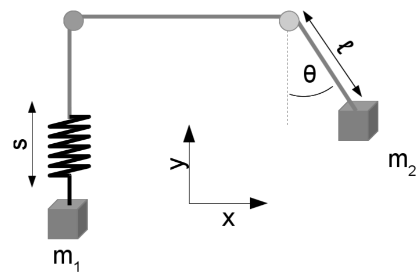
The determinant is $\omega^2(\omega^2 - \omega_0^2)\left(\omega^2 - \left(\frac{2+\lambda}{\lambda}\right)\omega_0^2\right)$, and therefore the normal frequencies are

$$\omega_1 = 0, \quad \omega_2 = \omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{2+\lambda}{\lambda}}\omega_0 = \sqrt{\frac{k}{m} + \frac{2k}{M}}.$$

(d) For the case of ω_1, the zero frequency just means that all three masses have identical motion; that is, the molecule is drifting through space. This type of motion is possible because there are no forces anchoring the molecule to a particular position in space (in contrast, the other systems we have studied, such as two carts with three springs and the double pendulum, always have at least one fixed reference point...motion away from these points produces a restoring force.) Since neither spring expands or contracts in this mode, there is zero oscillation frequency.

Figure 1: Solution to problem 1

2 Machine of Atwood with a spring



Two masses m_1 en m_2 are connected by a mass-less line of length L which runs without friction over two pulleys . Between mass 1 and the line there is an ideal spring with a spring constant k; the motion of mass 1 is limited to the vertical y-direction. Mass 2, on the other hand, can move freely in the plane of the drawing. The extension s of the spring is measured from its equilibrium point, not from the point of zero force on the spring.

You may assume that the motions occur in such a manner that neither spring nor masses ever jump over the pulleys. Please note that part *e)* is on the next page. [total: 17 pt]

a) Calculate the potential energy U and the kinetic energy T of this system in the given coordinates. (5 point)

b) Calculate the Lagrangian $\mathcal{L}$ and the generalized moments p_ℓ, p_s en p_θ. (4 point)

c) From this, find the Hamiltonian $\mathcal{H}(\ell, s, \theta, p_\ell, p_s, p_\theta)$ and show that it is equal to $T + U$. (5 point)

d) Which conserved quantity in this system is *not* connected to an ignorable coordinate? (1 point)

e) The Hamilton Formalism ($\mathcal{L}$, $\mathcal{H}$) and the laws of Newton are equivalent, yet some systems are easy to solve in one of them, while difficult in the other. Give one example each – including a short(!) argument for your choices – where (i) it is better to use the Hamilton formalism and (ii) where directly using the laws of Newton is the better choice. (2 points)

2.1 Solution

a) Voor de potengiële energie V berekenen wij zwartekrachtspotentiaal en veerpotentiaal:

$$\begin{aligned}
U &= \frac{1}{2}k\left(s+s_{\text{o}}\right)^2 - m_1 g\left(L-\ell+\left(s+s_{\text{o}}\right)\right) - m_2 g\ell\cos\theta \\
&= \frac{1}{2}ks^2 + ks_{\text{o}}s + m_1 g\ell - m_1 gs - m_2 g\ell\cos\theta + \text{const} \\
&= \frac{1}{2}ks^2 + m_1 g\ell - m_2 g\ell\cos\theta \qquad \left(s_{\text{o}} \overset{!}{=} m_1 g/k\right)
\end{aligned}$$

Voor de kinetische energie K vinden wij:

$$\begin{aligned}
K &= \frac{1}{2}m_1\left(-\dot{\ell}+\dot{s}\right)^2 + \frac{1}{2}m_2\left(v_{x,2}^2+v_{y,2}^2\right) \\
&= \frac{1}{2}m_1\left(\dot{\ell}^2 - 2\dot{\ell}\dot{s} + \dot{s}^2\right) + \frac{1}{2}m_2\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right)
\end{aligned}$$

b) De Lagrangiaan is gegven door $\mathcal{L} = K - V$; een gegeneralizeerde impuls is gedefinieerd als $\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$, ofwel $\frac{\partial K}{\partial\dot{q}_i}$ omdat V in het algemeen niet van de snelheid afhangt:

$$\begin{aligned}
p_\ell &= \frac{\partial\mathcal{L}}{\partial\dot{\ell}} &&= m_1\dot{\ell} - m_1\dot{s} + m_s\dot{\ell} \\
p_s &= \frac{\partial\mathcal{L}}{\partial\dot{s}} &&= -m_1\dot{\ell} + m_2\dot{s} \\
p_\theta &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}} &&= m_2\ell^2\dot{\theta}
\end{aligned} \tag{1}$$

c) De algemeene definitie van het Hamiltoniaan is $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$. Met behulp van 2 zien wij dat dit inderdaad overeenkomt met $T + V$. De Hamiltoniaan is:

$$\mathcal{H} = \frac{1}{2m_1} p_s^2 + \frac{1}{2m_2} \left((p_s + p_\ell)^2 + p_\theta^2 \right) + \frac{1}{2} k s^2 + m_1 g \ell - m_2 g \ell \cos\theta$$

d) $\mathcal{H} = T + V$ is niet tijdsafhankelijk, dus is de totaale energie behouden (en verschijnt niet als een negeerbare coördinaat)

e) Systeemen met bekende, constante krachten (vrije val) zijn het makkelijkst optelossen met Newtonse mechanica. Hamilton zou hier allen tot meer rekenwerk leiden. Systemeen waar de krachten niet bekend of van de snelheid / positie afhankelijk zijn los je beter met Lagrange of Hamilton (dubble pendulum, machine van atwood).

3 One dimensional map

Consider the one-dimensional map

$$x_{n+1} = \begin{cases} 3x_n & \text{for } 0 \le x_n \le 1/3 \\ \frac{3}{2}(1 - x_n) & \text{for } 1/3 < x_n \le 1 \end{cases}$$

[total: 13 pt]

(a) Find the fixed points of this map and analyze their stability. (2+2 points)

(b) Sketch the second iterate, x_{n+2}, of the map. Find the location of the period two orbits, and analyze their stability. (5+4 points)

3.1 Solution

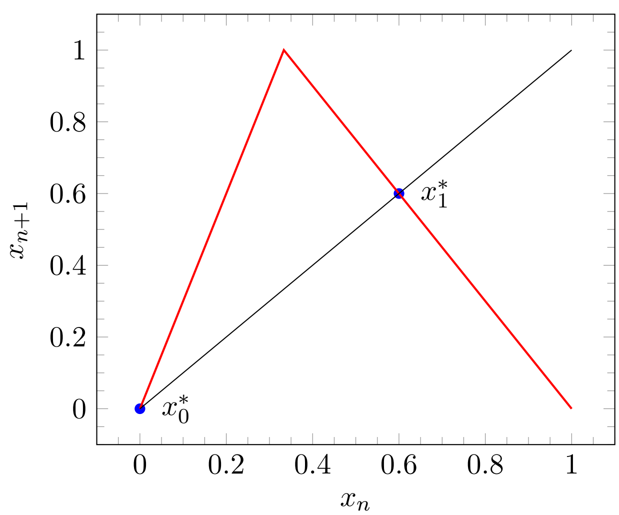


Figure 2: Skewed tent map. The fixed points are 0 and 3/5; as the slopes are 3 and −3/2, $|f'(x^*)| > 1$ and both fixed points are unstable.

For the solution, see figures 2 and 3.

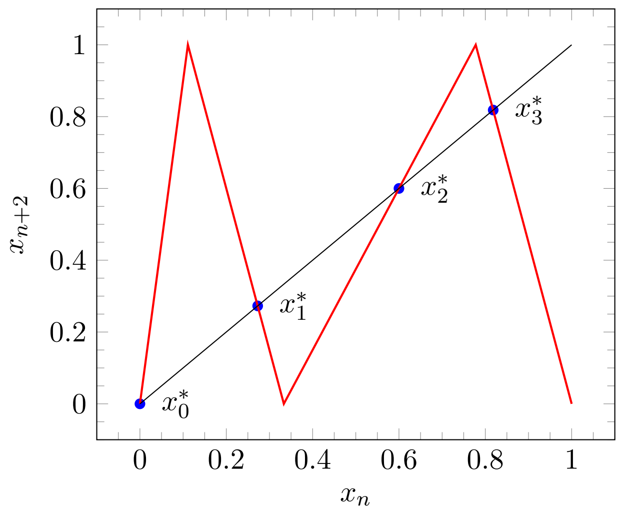
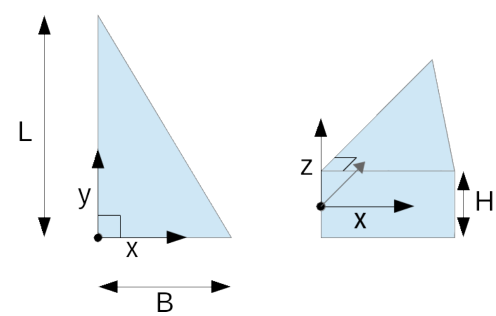


Figure 3: Skewed tent map, second iterate. As the tent has its apex at 1/3, the second iterate consists of a (compressed) copy between 0 and 1/3, and an inverted (compressed) copy between 1/3 and 1. The fixed points are 0, 3/7, 3/5, and 9/11. The slopes of the segments are 9, −9/2, 9/4, and −9/2; therefore all fixed points, again, are unstable.

4 Rigid Body Rotations and Chandler Wobble

We are considering rotational motion of a rigid body with homogeneous mass distribution (mass M, density ρ). Let us start with a triangular prism where $H < B < L$. [total: 15 pt]



(a) Calculate the inertial tensor **I** of the prism relative to the centre of rotation and the axes shown in the figure! (4 points)

(b) One of the given axes is a principal axis ("hoofdas"). Identify this principal axis from the found inertial tensor and symmetry considerations, and in doing so define the term "principal axis". Why do you find a principle axis for this body though it does not possess rotational symmetry? (4 points)

For the second part of this problem we consider a rigid body with only two different principal moments $\lambda_1 = \lambda_2 \neq \lambda_3$. Our *body frame*, as per definition, is fixed in the centre of mass and its axis are pointing along the principal axis of the body.

(c) Write out the three Euler Equations for this rigid body without external torque. (2 points)

(d) The rotation of Earth is well described by the Euler Equations that you have found ($\hat{e}_3$ is the rotational axis of earth, $\lambda_3 \approx 306/305\lambda_1$). The Euler Equations suggest that the direction of the angular velocity $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ of Earth changes in time. Find the period of this "Chandler Wobble". (3 points)

(e) In reality, the Chandler Wobble has a period of 433 days, which is not the period you have found. Give two potential explanations that could explain this discrepancy – other than that we used the wrong principal axes or principal moments. (2 points)

4.1 Solution

(a) We can calculate the elements of the inertial tensor from the formula:

$$I = \int dx \int dy \int dz\ \rho(x,y,z) \left(\left(x^2+y^2+z^2\right) \mathbf{E}_3 - \begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix} \right)$$

The integration boundaries are $(-\frac{H}{2}, \frac{H}{2})$ for z, $(0, L)$ for y and $(0, B(1-\frac{y}{L}))$ for x[1].

For the moments, we need to calculate the integrals over x^2, y^2, and z^2:

$$\begin{aligned}
\rho \iiint x^2\ dV &= \rho \int_{-H/2}^{H/2} dz \int_0^L dy \int_0^{B(1-y/L)} x^2 dx = \\
&= H\rho \int_0^L \frac{1}{3} x^3 \bigg|_0^{B(1-y/L)} = \\
&= \frac{1}{3} HB^3 \rho \underbrace{\int_0^L (1-y/L)^3 dy}_{v=1-y/L \Rightarrow L\int_0^1 v^3 dv} = \frac{1}{12} HB^3 L\rho = \frac{1}{6} B^2 M \\
\rho \iiint y^2\ dV &= \rho H \int_0^L y^2 B(1-\frac{y}{L}) dy = \\
&= \frac{1}{12} HBL^3 \rho = \frac{1}{6} L^2 M \\
\rho \iiint z^2\ dV &= \rho \int_{-H/2}^{H/2} z^2 dz \int_0^L B(1-y/L) dy = \\
&= \frac{1}{2} BL\rho\ \frac{1}{3} z^3 \bigg|_{-H/2}^{H/2} = \frac{1}{12} MH^2
\end{aligned}$$

For the products, we can see that all integrals over z will vanish due to symmetry, so the only product of interest is:

$$\iiint -\rho xy dV = -\frac{1}{24} \rho HL^2 B^2 = -\frac{1}{12} MLB.$$

This leads us to:

$$\mathbf{I} = \frac{1}{12} M \begin{pmatrix} 2L^2 + H^2 & -LB & 0 \\ -LB & 2B^2 + H^2 & 0 \\ 0 & 0 & 2B^2 + 2L^2 \end{pmatrix}.$$

(b) The z-axis is a principal axis, as for a rotation around this axis, that is for $\vec{\omega} = \omega_3 \hat{e}_3$, the angular momentum is parallel to the angular velocity. In this case, $\vec{L} = \mathbf{I}\vec{\omega} = \frac{1}{12} M(2B^2 + L^2)\omega_3 \hat{e}_3 \parallel \vec{\omega}$.

Rotational Symmetry is not a requirement for the existence of principal axes. *Every* rigid body has three principal axes, so it is not surprising to find them in this example. Furthermore, there are symmetries related to $\hat{z}$, namely that the xy-plane is a (mirror)symmetry plane of the body.

[1] or, of course, $(0, B)$ for x and $(0, L(1-x/B))$ for y. Also, note that the order of integration is now important for x and y!

(c) The form of Newton's law we are looking for is $\vec{F} = \frac{d\vec{p}}{dt}$. For a rotation, this becomes $\vec{\Gamma} = \frac{d\vec{L}}{dt}$, with $\vec{\Gamma}$ the torque acting on the body, and $\frac{d\vec{L}}{dt}$ the change of the angular momentum *in an inertial system.* To derive the Euler equation(s), we need to transform this into the (non-inertial) body frame, where the time derivative gains an extra term:

$$\vec{\Gamma} = \dot{\vec{L}} + \vec{\omega} \times \vec{L},$$

where the "dot" now refers to a simple time derivative in the non-inertial coordinate system. As the the coordinate axes of the body frame are principal axes, we know that $\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$. Writing out the cross product, we obtain:

$$\begin{aligned}
\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2\omega_3 &= \Gamma_1 \\
\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 &= \Gamma_2 \\
\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1\omega_2 &= \Gamma_3
\end{aligned}$$

Without external torque and for $\lambda_1 = \lambda_2$ we get

$$\begin{aligned}
\lambda_1\dot{\omega}_1 - (\lambda_1 - \lambda_3)\,\omega_2\omega_3 &= 0 \\
\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 &= 0 \\
\lambda_3\dot{\omega}_3 &= 0
\end{aligned}$$

(d) We see that $\dot{\omega}_3 = 0$ and thus $\omega_3 = \text{const}$:

$$\begin{aligned}
\dot{\omega}_1 - \underbrace{\left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3\right)}_{\Omega_b}\omega_2 &= 0 \\
\dot{\omega}_2 - \left(\frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\right)\omega_1 &= 0
\end{aligned}$$

You can now either use the trick of turning the two equations into a single, complex differential equation by using $\omega_1 + i\omega_2 = \eta$ and therefore $\dot{\eta} = -i\Omega_b\eta$, or by directly trying the test solutions $\omega_1 = \omega_\text{o}\cos\Omega_b t$ and $\omega_2 = -\omega_\text{o}\sin\Omega_b t$.

This describes a circular "wobble" of $\vec{\omega}$ around $\hat{e}_3$ with angular frequency $\Omega_b = \frac{1}{305}\omega_3$. As the latter is the rotation of earth (period roughly 1 day), the period of the Chandler Wobble is 305 days.

(e) The two assumptions that are violated are: (1) there are external (gravitational) torques acting on earth, and (2) Earth is *not* a rigid body (remember the tides?). *[NB:]* no points were given for pointing out that earth is not a perfect sphere (assumed by having more than one principal moment), suggesting different principal moments or axes (explicitly stated in question), or invoking general relativity.