Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Re-take Exam, March 12, 2015

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Tip: Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Coupled oscillations

Two pendula, each of length ℓ and carrying a mass m at the end, are coupled by a spring (see figure). When the pendula make angles θ_1 and θ_2 with respect to the downward vertical line, which the gravitational acceleration g points towards, the potential energy of the coupling spring is given by

$$\frac{k\ell^2}{2}(\sin\theta_1 - \sin\theta_2)^2.$$

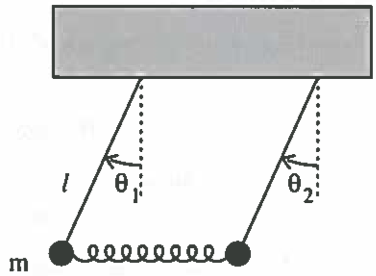

Figure 1: Two pendula, coupled by a spring

(total: 24 points)

(a) Write down the Lagrangian of the system, and derive the equation of motion for the two angles θ_1 and θ_2. (4 points)

(b) To tidy things up a bit, use the following notations:

$$\omega_0^2 = g/\ell, \quad \epsilon\omega_0^2 = k/m \quad \text{and} \quad \epsilon = k\ell/(mg).$$

In terms of these new variables, linearize the equations of motion in (a) for small θ_1 and θ_2. (4 points)

(c) From the linearized equations of motion for θ_1 and θ_2, identify the normal modes (4 points) and normal frequencies (4 points) of small oscillations. What sort of coupled motions of the pendula do these normal modes correspond to? (4 points)

(d) Assume that the amplitudes for the normal modes are real and are equal (equal to a). Schematically draw $\theta_1(t)$ and $\theta_2(t)$ under this assumption for the case of weak spring coupling, corresponding to $\epsilon \ll 1$. (4 points)

2 Constrained Lagrangian

A cylinder of mass m and radius R rolls down a wedge of angle α fixed on a horizontal surface (see Fig. 2), starting from rest at the top of the wedge at time $t = 0$. In order to describe the equation of motion for the system, we adopt two co-ordinates at time t: (i) $x(t)$ the distance of the center of the cylinder from the top of the wedge and (ii) $\theta(t)$ the rotation of a point on the periphery of the cylinder around its center. **(total: 26 points)**

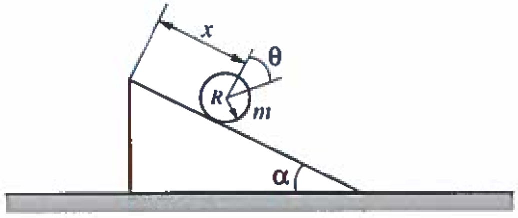

Figure 2: Cylinder rolling down a stationary wedge.

(a) Draw the free body diagram of the cylinder clearly showing all the forces acting on the cylinder. (4 points)

(b) Use Newton's laws to write down equations for $\ddot{\theta}$ and $\ddot{x}$. (5 points)

We will next address the problem using constrained Lagrangian (note: solving the problem without constraints will get you no points!).

(c) Relate the two variables θ and x in the form of $f(\theta, x) = 0$. This should be your constraint of motion. (5 points)

(d) Express the Lagrangian $\mathcal{L}$ in terms of θ and x. (3 points)

(e) Write down the Lagrangian equations of motion using the Lagrange multiplier. What is the physical meaning of of the Lagrange multiplier? (3+2 points)

(f) What would be the velocity of the cylinder when its centre comes down by a height H from its initial position? (4 points)

3 Non-inertial frame

Consider a frictionless puck on a horizontal turntable that is rotating counterclockwise (i.e., around the z-axis) with angular velocity Ω. **(total: 25 points)**

(a) Write down Newton's second law for the coordinates x and y of the puck as seen by me, standing on the turntable (ignore the earth's rotation). (5 points)

(b) Solve the two equations by the trick of writing $\eta = x + iy$ and guessing a solution of the form $\eta = e^{i\alpha t}$. Write down the general solution. (6 points)

(c) At time $t = 0$, I push the puck from position $\vec{r}_0 = (x_0, 0)$ with velocity $v_0 = (v_{x0}, v_{y0})$. Show that

$$x(t) = (x_0 + v_{x0}t)\cos\Omega t + (v_{y0} + \Omega x_0)t\sin\Omega t$$

and

$$y(t) = (x_0 + v_{x0}t)\sin\Omega t + (v_{y0} + \Omega x_0)t\cos\Omega t.$$

(4+4 points)

(d) Describe and sketch the behaviour of the puck for large values of t. (6 points)

4 Stanford (Torus) in Space

A common design for a future space station in both science and science fiction is based on a rotating "Stanford Torus"; the radius of the ring section is a, and the distance from the centre of mass to the centre of the ring is c. To simplify the math, we assume a homogeneous mass distribution in the ring (total mass M) and neglect any contributions of the spokes or structures in the centre. **(total: 25 points)**

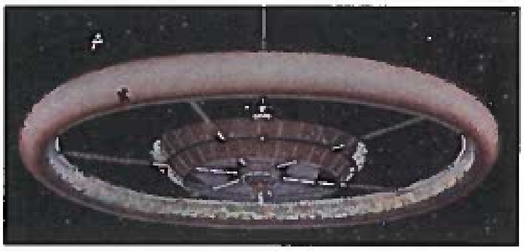

Figure 3: A "Stanford Torus" space station design, external view, ca 1975 (NASA Ames Research Centre)

(a) Find the principal axes of this space station, and calculate the three principal moments of inertia. (Hint: check the definitions in the formula section; if you cannot calculate the moments of inertia, explain how many distinct moments of inertia you expect in this case and continue the calculations denoting them λ_1, λ_2, etc.) (8 points)

(b) On the outer rim of the station there are 4 equidistant rocket motors used to slow down or speed up the rotation around $\hat{z}$, as needed. How fast does the station need to rotate so that the centrifugal acceleration at the rim is 1 g? How long (order of magnitude is enough!) will it take to reach this angular velocity? (outer diameter $D = 2500\,\text{m}$, ring diameter $d = 250\,\text{m}$, $M = 1 \times 10^{10}\,\text{kg}$. $F = 175\,\text{kN}$ per rocket) (5 points)

(c) Due to a maintenance error, the rockets did not only induce rotation around the z-axis (or $\hat{e}_3$), but unfortunately also induced a momentary small torque around a perpendicular axis (let us assume it was around $\hat{e}_1$). Use the Euler equations to analyze what happens to the space station and find the angular velocities $\omega_1, \omega_2, \omega_3$ as function of time t, starting with $\omega_2(0) = 0$ en $\omega_3(0) \gg \omega_1(0)$. (8 points)

(d) In reality, the Stanford Torus is not a rigid body. What sort of stress would tangentially aligned rockets cause on the torus, and where in the stress tensor would you find them? Consequently, which strain will this cause in the torus? (4 points)

Handige formules

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\arcsin(\gamma)) = \sqrt{1-\gamma^2} \qquad \cos(\pi+\alpha) = -\cos\alpha \qquad \sin(\pi+\beta) = -\sin\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)) \qquad \sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}}$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1) \quad \mu = \frac{m_1 m_2}{m_1+m_2} \quad \frac{c_1}{1+\epsilon_1\cos(\phi+\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi+\delta_2)}$$

$$U_{\text{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\text{rel}} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad J_{ij} = I_{ij}^{\text{cm}} + m\left(\|\vec{a}\|^2\delta_{ij} - a_i a_j\right)$$

$$\mathbf{I} = \iiint dV\, \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} \qquad \vec{F}(d\vec{A}) = \mathbf{\Sigma}d\vec{A} \quad \mathbf{\Sigma} = \alpha e\mathbf{1} + \beta\mathbf{E}', \; e = \frac{1}{3}\text{tr}\mathbf{E}$$

$$\left(\frac{dQ}{dt}\right)_{S_o} = \dot{Q} + \Omega\times Q \qquad \left(\frac{d^2Q}{dt^2}\right)_{S_o} = \ddot{Q} + 2\Omega\times\dot{Q} + \Omega\times(\Omega\times Q)$$

$$\mathcal{L} = T - V \qquad \frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial q} + \lambda\frac{\partial f(q)}{\partial q} = \frac{d}{dt}\frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial\dot{q}} \qquad f(q_1,q_2,\cdots,q_N) \overset{!}{=} \text{const}$$

$$\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = T + V \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \quad \dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \qquad \dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma}$$

$$\lambda\vec{\omega} = \mathbf{I}\vec{\omega} \qquad (\mathbf{I}-\lambda\mathbf{E}_3)\vec{\omega} = 0 \qquad \det(\mathbf{I}-\lambda\mathbf{E}_3) = 0 \qquad \mathbf{E}_3 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$\int_0^{2\pi}\sin^n(x)\cos^m(x)dx = 0$ for odd m or n. $\qquad \int_0^{2\pi}\cos^2(x)dx = \int_0^{2\pi}\sin^2(x)dx = \pi$

$$\arctan 0 = 0 \qquad \arctan\pm\infty = \pm\tfrac{1}{2}\pi$$

$$f(x^*) = x^*, \qquad f(x_a) = x_b \wedge f(x_b) = x_a \Rightarrow f(f(x^*)) = x^*$$

useful coordinates for Tori: angle u around the z-axis, angle v from the centre of the ring (distance c from the z-axis), and distance r' from the centre of the ring:

$$\begin{aligned} x &= (c + r'\cos v)\cos u \\ y &= (c + r'\cos v)\sin u \\ z &= r'\sin v \end{aligned}$$

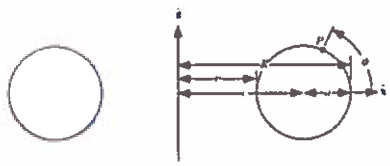

Figure 4: Ring Torus

$$V = \iint_{\text{torus}} dxdydz = \int_0^a\int_0^{2\pi}\int_0^{2\pi} r'(c + r'\cos v)\,dr'dudv = 2\pi^2a^2c$$