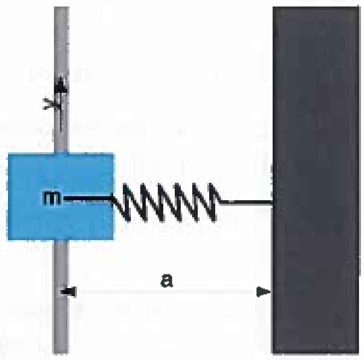
Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Midterm, 15 December 2016

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Tip: Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 All Oscillators are Harmonic – almost

We are considering a slightly idealized, one dimensional motion of a block of mass m along a thin metal rail that goes through the center of the block. A massless spring of stiffness k is fastened to the wall, a distance a from the rail, and to the center of mass of the block. The resting length of the spring, ℓ_o, is not necessarily equal to a. Gravity acts in the $-\hat{y}$ direction, and there is no friction. Set the orgin such that at $y = 0$ the spring is horizontal. [total: 10 points]



(a) Calculate the total force $F(y)$ acting on the mass on the rail in the y direction. (2 points)

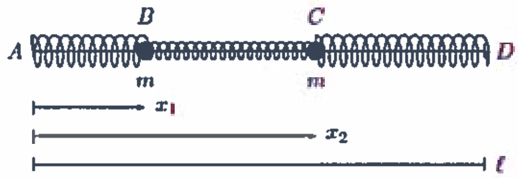
(b) Find the equation of motion for this system, which should contain a constant term, a harmonic term $\sim \omega_0^2 y$ and a non-linear term. Expand the non-linear term to third order around $y = 0$. (3 points)

You will of course have noticed that there is a problem with the expansion around $y = 0$: there is no minimum of the potential at that point! To remedy this, we now turn the system on its side so that gravity no longer contributes to the motion of the mass. So please set $g = 0$ before continuing.

(c) Find the *stationary point(s)* of the mass m, that is those points where the total force on the mass is zero and it can remain at rest, for the two cases $\ell_o > a$ and $\ell_o < a$. How many – if any – such points are there, and are they stable or unstable? *Note:* Do not use the Taylor expansion for this! (3 points)

(d) For $\ell_o = a$ there is an anomaly. Sketch the potential energy of the mass for the cases that ℓ_o is slightly smaller, equal, and slightly larger than a. Argue why or why not a small oscillation would be harmonic in these three cases. *Note:* Now you are allowed to use the Taylor expansion! (2 points)

2 Lagrange of a Springy System



Three massless perfectly elastic springs AB, BC and CD are attached as shown in the Fig.1. The spring ends in A and D are fixed. Two objects of mass m are located where the springs join in B and C. The springs AB and CD have natural lengths a and spring constant k, while BC has natural length b and spring constant k_1. The distance l between the fixed end points AD is greater than the total natural lengths of the springs: $l > 2a + b$. [total: 10 points]

(a) Using x_1 and x_2, as shown in the diagram for the coordinates of the two masses, construct the Lagrangian. (1 point)

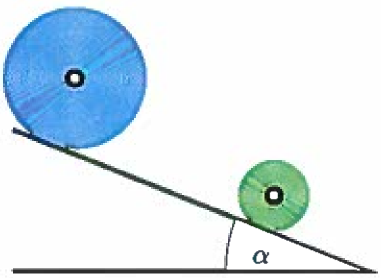
(b) Determine the equations of motion from the Lagrangian. (2 points)

(c) Set $\ddot{x}_1$ and $\ddot{x}_2$ to zero to determine the equilibrium positions of the masses (i.e. positions where the system stays fixed). (2 points)

(d) Find the two eigenfrequencies of the system, ω_1 and ω_2, by solving the equations of motion. (2 points)

(e) The solutions are quasi-periodic. Describe the motion of the masses for each one of the two frequencies ω_1 and ω_2 that you found in part d. (3 points)

3 Paper Reel on the Loose



After papermaking, a long sheet of paper (total length L) is rolled onto a cylindrical *reel* – a bit like the one you find inside a roll of sticky tape, but much wider – for shipping and later processing. The reel has an outer diameter of D, including the paper on it; the total mass of the paper is M. For the calculation, you may ignore the size and mass of the inner part of the reel.

Due to inattention of one of the apprentices, one of these reels falls off during loading; the paper gets caught on the edge of a ramp, and the reels start rolling (*thus not sliding!*) down a long incline of angle α. Assume that the paper does not rip and the slowly unrolling reel stays cylindrical in shape. Do not forget that although we assume that the unrolled part has negligible thickness, it still has mass, uniformly distributed along the length of the unrolled part. [total: 10 points]

(a) Are the energies and momenta of the reel conserved? Include an argument in your answer. (1+1+2 points)

(b) Use conservation of energy to determine the velocity of the reel's centre when the diameter of the unrolled, cylindrical, part becomes $D/2$. (6 points)

Useful Formulas

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\arcsin(\gamma)) = \sqrt{1-\gamma^2} \qquad \cos(\pi+\alpha) = -\cos\alpha \qquad \sin(\pi+\beta) = -\sin\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right) \qquad \sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$$

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}}$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}\left(\epsilon^2-1\right) \quad \mu = \frac{m_1 m_2}{m_1+m_2} \quad \frac{c_1}{1+\epsilon_1\cos(\phi+\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi+\delta_2)}$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_o} = \dot{\mathbf{Q}} + \mathbf{\Omega}\times\mathbf{Q} \qquad \left(\frac{d^2\mathbf{Q}}{dt^2}\right)_{S_o} = \ddot{\mathbf{Q}} + 2\mathbf{\Omega}\times\dot{\mathbf{Q}} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{Q})$$

$$f_{\mathrm{cstr},j}(q_i) = 0 \qquad \mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L}$$

$$\mathcal{L}(q_i,\dot{q}_i,t) = T - V \qquad \frac{\partial\mathcal{L}}{\partial q_i} + \sum_j \lambda_j \frac{\partial f_{\mathrm{cstr},j}}{\partial q_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$$

$$U_{\mathrm{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\mathrm{rel}} = \tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r) = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \mathcal{O}(x^2) \qquad T_{\mathrm{cyl}} = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 \qquad w(\phi) = u(\phi) - C$$

$$f(a) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n\,; \quad f^{(n)} = \frac{d^n f}{dx^n}$$

$$I_{\mathrm{cylinder}} = \frac{1}{2}MR^2$$