Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Retake Exam, April 20, 2017

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Hamiltonian

A mass m is connected to a moving piston by a horizontal spring with spring constant k and relaxed length l_0. The piston is arranged to move back and forth with position $X_{\text{wall}} = A\sin\omega t$. Let z measure the extension of the spring from its equilibrium length. **(total: 20 points)**

(a) Determine the Hamiltonian $\mathcal{H}$ in terms of z and its conjugate momentum (*Hint: take the location of the piston at t =0 as origin of your coordinate system*) (6 points)

(b) Write down Hamilton's equations. (5 points)

(c) Is $\mathcal{H}$ the total energy? (discuss the general case and then refer to this particular problem) (3 points)

(d) Is $\mathcal{H}$ conserved? (discuss the general case and then refer to this particular problem) (3 points)

(e) Is the total energy conserved? (3 points)

2 Fictitiously Down-Under

Qantas flight QF575 flies from Sydney, New South Wales, Australia, to Perth, Western Australia. The place moves with constant speed v and at constant height h in a westerly direction (latitude 35° south). **(total: 25 points)**

a) Make a sketch of all forces – real and fictitious – acting upon the plane on its way to Perth. Indicate the direction of the angular velocity of earth $\vec{\Omega}$. *N.B. :* As you surely know, Australia is in the southern hemisphere! (4 points)

b) We now move to a coordinate system in which z points up (defined as opposing the force of gravity acting on the plane at rest), x points to the north, and y to the west. Calculate the components of the coriolis force in these coordinates. (8 points)

Shortly after take-off, the plane has to be re-routed to Melbourne due to an unexpectely strong storm; the plane now flies with the same speed v and height h, but in a south-westerly direction.

c) Make a sketch of all fictitious forces acting on the plane on its new course. Use the coordinates introduced in part b. [*Hint:* $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$] (4 points)

d) A travelling physicist has put up a bob (a weight on a string) in the plane. This bob, hanging from the ceiling of the economy class cabin, oscillates around a zero-position (with $-\vec{z}_o$ pointing downwards) with a period T_o. Both zero-position and period were determined while the plane was standing still in Sydney. Explain whether and how these two parameters would change if:

i) earth was not rotating at all? (2 points)

ii) the plane follows its original course towards the west? (3 points)

iii) the plane follows its new course towards the south-west? (3 points)

The airplane was slightly damaged in the storm and is grounded for repairs in Melbourne. Meanwhile, the bob continues to oscillate like a pendulum (ignore the friction). Slowly the plane in which it oscillates changes.

iv) What is the name of this type of pendulum, and in which direction does the plane of oscillation turn? (3 points)

3 Kepler Orbits of Higher Order Potentials

Consider two particles (reduced mass μ) orbiting each other under a central force with potential energy $U(r) = kr^n$ with $kn > 0$. **(total: 25 points)**

(a) Explain what the condition $kn > 0$ tells us about the nature of the force. (3 points)

(b) With given angular momentum ℓ, sketch the effective potential energy U_{eff} for the cases $n = 2, -1$ and -3; make sure to sketch each term of the potential as well as the total potential, and indicate whether the potential has an extremum. (3+3+3 points)

(c) Find the radius at which the particle (with given angular momentum ℓ) can orbit at a fixed radius. For which value of n is this circular orbit stable (do your sketches confirm this conclusion)? (3+2 points)

(d) For the stable case show that the period of small oscillations about the circular orbit is $\tau_{\text{osc}} = \tau_{\text{orb}}/\sqrt{n+2}$. (4 points)

(e) Argue that if $\sqrt{n+2}$ is a rational number then these orbits are closed. (4 points)

4 Tumbling Rotations

In this question, we will deal with the rotation of rigid bodies in three dimensions. Take a rectangular box of mass M and homogenous density ρ. As shown in the figure on the right, the box has dimensions $h < b < l$ and is rotating around a point along one of its shortest edges. (total: 20 points)

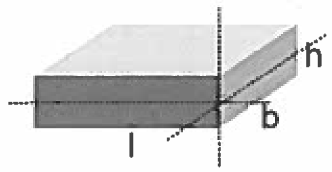

Figure 1: Rectangular Prism with homogeneous mass distribution. The origin is in the centre of one of the four shortest edges.

(a) Calculate the inertial tensor I around the given axes and origin. (6 points)

(b) The inertial tensor consists of *moments of inertia* and *products of inertia*. Explain these two terms and give the physical meaning of the two elements of the inertial tensor I_{zy} and I_{yz} for a rotation $\vec{\omega}$ around an arbitrary axis. Argue from symmetry whether one of those two elements has to always be larger than the other. (3 points)

(c) Around which of the three axes given in the figure can the body only rotate if there is an external torque? Give an explanation why this is so. (3 points)

(d) Are any of the axes given in the figure *principal axes* of the body? Start by explaining what a "principal axis" is! (2 points)

Let us now consider rotations of the body occuring without an external torque (free rotations):

(e) Through which point (origin) do we find the principal axes which correspond to the smallest *principal moments* λ_i? What is a "principal moment" and which direction will the principal axes have in this case? (3 points)

Useful Formulas

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\arcsin(\gamma)) = \sqrt{1-\gamma^2} \qquad \cos(\pi+\alpha) = -\cos\alpha \qquad \sin(\pi+\beta) = -\sin\beta$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)) \qquad \sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}}$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2-1) \quad \mu = \frac{m_1 m_2}{m_1+m_2} \quad \frac{c_1}{1+\epsilon_1\cos(\phi+\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi+\delta_2)}$$

$$U_{\text{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\text{rel}} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad J_{ij} = I^{\text{cm}}_{ij} + m\left(||\vec{a}||^2\delta_{ij} - a_i a_j\right)$$

$$\mathbf{I} = \iiint dV\ \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

$$\text{sphere: } I_{zz} = \frac{2}{5}mr^2 \qquad \text{cylinder: } I_{zz} = \frac{1}{2}mr^2 \qquad \text{hoop: } I_{zz} = mr^2$$

$$\left(\frac{dQ}{dt}\right)_{S_o} = \dot{Q} + \Omega\times Q \qquad \left(\frac{d^2Q}{dt^2}\right)_{S_o} = \ddot{Q} + 2\Omega\times\dot{Q} + \Omega\times(\Omega\times Q)$$

$$\mathcal{L} = T - V \qquad \frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial\dot{q}} = 0$$

$$\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = T + U \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \quad \dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \qquad \dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma}$$

$$\lambda\vec{\omega} = \mathbf{I}\vec{\omega} \qquad (\mathbf{I}-\lambda\mathbf{E_3})\vec{\omega} = 0 \qquad \det(\mathbf{I}-\lambda\mathbf{E_3}) = 0 \qquad \mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_i(d\vec{A}) = \sum_j \sigma_{ij} dA_j \qquad \vec{F}(d\vec{A}) = \Sigma d\vec{A} = \Sigma\hat{n}dA$$

For a surface and a point on that surface: $S(x,y,z) = 0,\ P \in S: \quad \vec{n} = \nabla S$

$\int_0^{2\pi}\sin^n(x)\cos^m(x)dx = 0$ for odd m or n. $\qquad \int_0^{2\pi}\cos^2(x)dx = \int_0^{2\pi}\sin^2(x)dx = \pi$

$$\arctan 0 = 0 \qquad \arctan\pm\infty = \pm\tfrac{1}{2}\pi \qquad \det(\mathbf{K}-\omega^2\mathbf{M}) = 0$$

$$\sin 35^\circ = 0.574 \qquad \cos 35^\circ = 0.819 \qquad \tan 35^\circ = 0.700$$

$$f(x^*) = x^* \qquad f(x_a) = x_b \wedge f(x_b) = x_a \Rightarrow f(f(x^*)) = x^* \qquad x_t = x^* + \epsilon \Rightarrow \epsilon_{t+1} \approx f'(x^*)\epsilon_t$$