Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Final; February 1, 2018

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Tip: Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Hamiltonian in a Central Forcefield

A particle moves under the influence of a central force $\vec{F}(\vec{r},t) = -\frac{k}{r^2}e^{-\beta t}\hat{r}$, where k and β are positive and constant, t is time, and r is the distance of the particle to the origin. **(total: 15 points)**

a) Using the general definition of a Hamiltonian, find $\mathcal{H}$ of this system. (6 points)

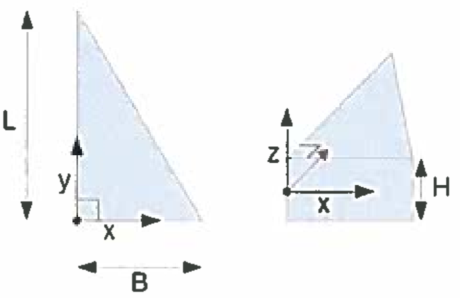
b) Use the Hamiltonian to find the equation of motion of the particle. (5 points)

c) Compare the Hamiltonian you calculated to the total energy. Why is it equal/not equal to the total energy $T+U$? (2 points)

d) Is the energy of the particle conserved? Use a simple example to discuss why (not). (2 points)

2 Rigid Body Rotations and Chandler Wobble

We are considering rotational motion of a rigid body with homogeneous mass distribution (mass M, density ρ). Let us start with a triangular prism where $H < B < L$. **(total: 22 points)**



(a) Calculate the inertial tensor $\mathbf{I}$ of the prism relative to the centre of rotation and the axes shown in the figure! (8 points)

(b) One of the given axes is a principal axis ("hoofdas"). Identify this principal axis from the found inertial tensor and symmetry considerations, and in doing so define the term "principal axis". Why do you find a principle axis for this body though it does not possess rotational symmetry? (4 points)

For the second part of this problem we consider a rigid body with only two different principal moments $\lambda_1 = \lambda_2 \neq \lambda_3$. Our *body frame*, as per definition, is fixed in the centre of mass and its axis are pointing along the principal axis of the body.

(c) Write out the three Euler Equations for this rigid body without external torque. (3 points)

(d) The rotation of Earth is well described by the Euler Equations that you have found ($\hat{e}_3$ is the rotational axis of earth, $\lambda_3 \approx 306/305\lambda_1$). The Euler Equations suggest that the direction of the angular velocity $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ of Earth changes in time. Find the period of this "Chandler Wobble". (4 points)

(e) In reality, the Chandler Wobble has a period of 433 days, which is not the period you have found. Give two potential explanations that could explain this discrepancy – other than that we used the wrong principal axes or principal moments. (3 points)

6cm

3 Coupled Oscillations

Two blocks and three springs are connected as shown in the figure. All motion is horizontal. When the blocks are at rest, all springs are unstretched. **(total: 18 points)**

(a) Choose x_1 and x_2 as generalized coordinates for the displacement from the equilibrium positions of the blocks with masses m_1 and m_2, and find the equation of motion for both blocks. (5 points)

(b) Combine the set of of motions into matrix form $\mathbf{M}\ddot{\vec{x}} = -\mathbf{K}\vec{x}$. (3 points)

From now on, use $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$ and $k_3 = 2k$.

(c) Find the frequencies of (small) oscillations of the modes. (6 points)

(d) Find the normal modes of this system. Describe the physical motions to which the normal modes correspond. (4 points)

4 Multiple Choice

The final questions for this exam are multiple choice (on a separate sheet). Please make sure to remove the staple *cleanly* before you hand in the answer sheet. [total: 5+2 bonus points]

Useful Formulas

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}} \qquad \text{hoop: } I_{zz} = mr^2$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}\left(\epsilon^2 - 1\right) \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad \frac{c_1}{1+\epsilon_1\cos(\phi+\delta_1)} = \frac{c_2}{1+\epsilon_2\cos(\phi+\delta_2)}$$

$$U_{\text{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\text{rel}} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad J_{ij} = I_{ij}^{\text{cm}} + m\left(||\vec{a}||^2\delta_{ij} - a_i a_j\right)$$

$$\mathbf{I} = \iiint dV\ \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

$$\mathcal{L} = T - V \qquad \frac{\partial\mathcal{L}\left(q(t),\dot{q}(t),t\right)}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}\left(q(t),\dot{q}(t),t\right)}{\partial\dot{q}} = 0$$

$$\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = T + U \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \quad \dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \quad \dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma} \qquad \frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \mathcal{O}(x^2)$$

$$\lambda\vec{\omega} = \mathbf{I}\vec{\omega} \qquad (\mathbf{I} - \lambda\mathbf{E_3})\vec{\omega} = 0 \qquad \det(\mathbf{I} - \lambda\mathbf{E_3}) = 0 \qquad \mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_i(d\vec{A}) = \sum_j \sigma_{ij}dA_j \qquad \vec{F}(d\vec{A}) = \mathbf{\Sigma}d\vec{A} = \mathbf{\Sigma}\hat{n}dA$$

For a surface and a point on that surface: $S(x,y,z) = 0,\ P \in S:\quad \vec{n} = \nabla S$

$\int_0^{2\pi}\sin^n(x)\cos^m(x)dx = 0$ for odd m or n. $\qquad \int_0^{2\pi}\cos^2(x)dx = \int_0^{2\pi}\sin^2(x)\ dx = \pi$

$$\arctan 0 = 0 \qquad \arctan \pm\infty = \pm\tfrac{1}{2}\pi \qquad \det(\mathbf{K} - \omega^2\mathbf{M}) = 0$$

$$f(x^*) = x^* \qquad f(x_a) = x_b \wedge f(x_b) = x_a \Rightarrow f(f(x^*)) = x^* \qquad x_t = x^* + \epsilon \Rightarrow \epsilon_{t+1} \approx f'(x^*)\epsilon_t$$

$$f^{(n)}(x) = \underbrace{f(f(\cdots f(x)\cdots))}_{\text{ntimes}}$$

Voortgezette Mechanica (NS-350B) **Final Exam Multiple-Choice Part**

No documents allowed. The use of electronic devices is forbidden.
Questions using the sign ♣ may have zero, one or several correct answers and are worth 3 points.
All other questions have a single correct answer and are worth 2 points.

Question 1 For small oscillations (linearized case), the motion of a coupled oscillator with n degrees of freedom can be written using $n \times n$ "mass" and "spring-constant" matrices:

$$\mathbf{M}\ddot{\vec{q}} = -\mathbf{K}\vec{q}.$$

We can find the normal frequencies ω associated with these small oscillations by solving:

(A) $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ (B) $\det(\mathbf{K} - \mathbf{M}) \cdot \vec{a} = \omega^2$ (C) $\omega^2 = \mathrm{diag}(\mathbf{K} \cdot \mathbf{M}^{-1})$
(D) $\omega_i = \frac{k_i}{m_i}$ (E) $\det(\mathbf{M} - \omega\mathbf{K}) = 0$

Question 2 The Euler Angles θ, ϕ, and ψ allow us to connect the Euler equation in the body frame to an inertial frame. It turns out that θ is moving in an effective potential, which causes a nodding motion called ...

(A) Chandler wobble. (B) free rotation. (C) steady precession.
(D) free precession. (E) nutation.

Question 3 ♣ A 'boost' in special relativity refers to ...

(A) ... the additional factor in the relativistic velocity-addition formula applied to the component parallel to the relative motion.

(B) ... the additional factor in the relativistic velocity-addition formula applied to the components perpendicular to the relative motion.

(C) ... any Lorentz transformation that leaves corresponding axes parallel.

(D) ... the change of length (length contraction) of an object perpendicular to its relative velocity, when measured from an inertial system at rest.

(E) ... the time delay Δt of two events which appear simultaneous in a frame at rest, when measured in a moving frame.

Question 4 ♣ We found that transverse motion on an infinite string leads to the wave equation. Which statements of the wave equation are correct?

(A) The wave equation has no solutions in more than two dimension.

(B) Standing waves do not travel along the string, their spatial and temporal functions are separated.

(C) For the sinusoidal f and g we find *standing waves* as a solution.

(D) All solutions to the wave equation on an infinite string are of the form $f(x+ct) + g(x-ct)$, travelling left (or right) on the string.

(E) The only solution to the wave equation on a *finite* string is a standing wave with one fixed frequency, depending on L.

Question 5 Starting from different initial condition, we can write the separation of the two resulting solutions as $|\Delta\varphi(t)| \sim e^{\lambda t}$. The coefficient λ is known as the

- (A) Control parameter
- (B) Feigenbaum constant
- (C) Liapunov exponent
- (D) Lagrange exponent
- (E) Poincaré exponent

Question 6 ♣ Coupled oscillations can be described in terms of *normal modes*. Which of these statements concerning normal modes are true?

- (A) Any periodic motion of a coupled oscillator can be written as a (weighted) sum of normal modes.
- (B) Every coupled oscillator has exactly one normal mode.
- (C) A normal mode is any motion in which all n coordinates oscillate sinusoidally with the same frequency ω.
- (D) In a symmetric case (same spring constant and masses), all n normal modes of a coupled oscillator have the same ω.
- (E) Any periodic motion of a coupled oscillator has to be exactly one of the normal modes.

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Question 7 ♣ Which of the following statements apply to the Inertia Tensor of an arbitrary rigid body?

- (A) The Inertia Tensor is a function of the shape of the body only and does not depend on the choice of origin or axes.
- (B) Every Inertia Tensor has three distinct eigenvalues or *principal moments* which are different from each other.
- (C) An axis of rotational symmetry is always a principal axis of a body.
- (D) Every Inertia Tensor can be diagonalized.
- (E) All elements of an Inertia Tensor I_{ij} have to be positive.

Question 8 A question about the order of magnitude of time dilation. Suppose a certain event happens at intervals of exactly one hour in an inertial frame moving at a steady speed of $300\,\mathrm{m\,s^{-1}}$. When measured from an inertial frame at rest, the interval will appear ...

- (A) ...longer by 2 nanoseconds.
- (B) ...exactly the same length in time, as both are inertial frames.
- (C) ...shorter by half a nanosecond.
- (D) ...longer by half a millisecond.
- (E) ...shorter by 2 milliseconds.