Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Final; February 1, 2018 – Model Solution

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Tip: Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Hamiltonian in a Central Forcefield

A particle moves under the influence of a central force $\vec{F}(\vec{r},t) = -\frac{k}{r^2}e^{-\beta t}\hat{r}$, where k and β are positive and constant, t is time, and r is the distance of the particle to the origin. **(total: 15 points)**

a) Using the general definition of a Hamiltonian, find $\mathcal{H}$ of this system. (6 points)

b) Use the Hamiltonian to find the equation of motion of the particle. (5 points)

c) Compare the Hamiltonian you calculated to the total energy. Why is it equal/not equal to the total energy $T+U$? (2 points)

d) Is the energy of the particle conserved? Use a simple example to discuss why (not). (2 points)

1.1 Solution

a) From our study of movement under a central force, we know that the problem is essentially two-dimensional, and we should use cylinder coordinates (r,ϕ). As the second question concerns the relationship between $\mathcal{H}$ and the total energy, we use the basic defintion

$$\mathcal{H} = \sum p_i \dot{q}_i - \mathcal{L}$$

to find the Hamiltonian. The Lagrangian is given by $T-V$, where $V = -\frac{k}{r}\exp(-\beta t)$ and $T = \frac{1}{2}\dot{\vec{r}}^2$:

$$\begin{aligned}\vec{r} = r\cos\phi\hat{x} + r\sin\phi\hat{y} \Rightarrow \dot{\vec{r}} &= \left(\dot{r}\cos\phi - r\dot{\phi}\sin\phi\right)\hat{x} + \left(\dot{r}\sin\phi + r\dot{\phi}\cos\phi\right)\hat{y}\\ \dot{\vec{r}}^2 &= \dot{r}^2\left(\sin^2\phi + \cos^2\phi\right) + r^2\dot{\phi}^2\left(\sin^2\phi + \cos^2\phi\right) + \cdots\\ & 2r\dot{r}\dot{\phi}\left(-\cos\phi\sin\phi + \sin\phi\cos\phi\right)\end{aligned}$$

We get the (expected) result $\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$. From this we can calculate the conjugate impulses $p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}$ and $p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi}$. The Hamiltonian is then defined as

$$\begin{aligned}\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L} &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) = \\ &= \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{k}{r}e^{-\beta t}\end{aligned} \tag{1}$$

b) To find the equation of motion, we need to solve the Hamilton Equations:

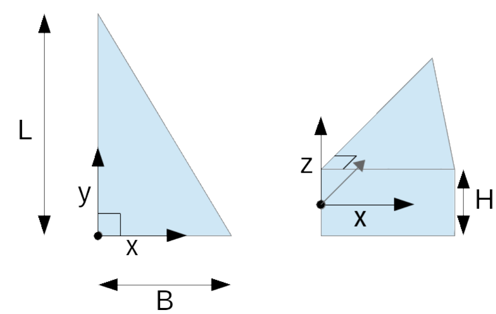
$$-\frac{\partial\mathcal{H}}{\partial r} = \dot{p}_r = \frac{p_\phi^2}{mr^3} - \frac{k}{r^2}e^{-\beta t} \qquad \frac{\partial\mathcal{H}}{\partial p_r} = \dot{r} = \frac{p_r}{m} \qquad \Rightarrow \begin{cases} \ddot{r} = \frac{1}{m}\dot{p}_r = \frac{p_\phi^2}{m^2r^3} - \frac{k}{mr^2}e^{-\beta t} \\ p_\phi = mr^2\dot{\phi} = \text{const} \end{cases}$$
$$-\frac{\partial\mathcal{H}}{\partial \phi} = \dot{p}_\phi = 0 \qquad \frac{\partial\mathcal{H}}{\partial p_\phi} = \dot{\phi} = \frac{p_\phi}{mr^2}$$

c) Despite the form and time dependence of the potential, the only requirement for $\mathcal{H}$ to represent the total energy is the choice of variables. Our variables are natural (*i.e.* not explicitly depending on time), therefore $\mathcal{H} = T + U$, as seen in equation 1.

d) The energy is not conserved, however it is only half right to point to the fact that the Hamiltonian is time dependent as proof. In principle the kinetic energy can make up a loss of potential energy! If we look at the Hamiltonian, however, we see that the potential is negative, and if we were to start with a bound orbit ($E < 0$), the kinetic energy would eventually have to become negative to conserve the total energy. As this is obviously not possible, the energy can not be conserved!

2 Rigid Body Rotations and Chandler Wobble

We are considering rotational motion of a rigid body with homogeneous mass distribution (mass M, density ρ). Let us start with a triangular prism where $H < B < L$. **(total: 22 points)**



(a) Calculate the inertial tensor **I** of the prism relative to the centre of rotation and the axes shown in the figure! (8 points)

(b) One of the given axes is a principal axis ("hoofdas"). Identify this principal axis from the found inertial tensor and symmetry considerations, and in doing so define the term "principal axis". Why do you find a principle axis for this body though it does not possess rotational symmetry? (4 points)

For the second part of this problem we consider a rigid body with only two different principal moments $\lambda_1 = \lambda_2 \neq \lambda_3$. Our *body frame*, as per definition, is fixed in the centre of mass and its axis are pointing along the principal axis of the body.

(c) Write out the three Euler Equations for this rigid body without external torque. (3 points)

(d) The rotation of Earth is well described by the Euler Equations that you have found ($\hat{e}_3$ is the rotational axis of earth, $\lambda_3 \approx 306/305\lambda_1$). The Euler Equations suggest that the direction of the angular velocity $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ of Earth changes in time. Find the period of this "Chandler Wobble". (4 points)

(e) In reality, the Chandler Wobble has a period of 433 days, which is not the period you have found. Give two potential explanations that could explain this discrepancy – other than that we used the wrong principal axes or principal moments. (3 points)

2.1 Solution

(a) We can calculate the elements of the inertial tensor from the formula:

$$I = \int dx \int dy \int dz\ \rho(x,y,z) \left(\left(x^2+y^2+z^2\right) \mathbf{E}_3 - \begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix} \right)$$

The integration boundaries are $(-\frac{H}{2}, \frac{H}{2})$ for z, $(0, L)$ for y and $(0, B(1-\frac{y}{L}))$ for x[1].
For the moments, we need to calculate the integrals over x^2, y^2, and z^2:

$$\begin{aligned}
\rho \iiint x^2\ dV &= \rho \int_{-H/2}^{H/2} dz \int_0^L dy \int_0^{B(1-y/L)} x^2 dx = \\
&= H\rho \int_0^L \frac{1}{3} x^3 \Big|_0^{B(1-y/L)} = \\
&= \frac{1}{3} HB^3 \rho \underbrace{\int_0^L (1-y/L)^3 dy}_{v=1-y/L \Rightarrow L\int_0^1 v^3 dv} = \frac{1}{12} HB^3 L\rho = \frac{1}{6} B^2 M \\
\rho \iiint y^2\ dV &= \rho H \int_0^L y^2 B(1-\frac{y}{L}) dy = \\
&= \frac{1}{12} HBL^3 \rho = \frac{1}{6} L^2 M \\
\rho \iiint z^2\ dV &= \rho \int_{-H/2}^{H/2} z^2 dz \int_0^L B(1-y/L) dy = \\
&= \frac{1}{2} BL\rho\, \frac{1}{3} z^3 \Big|_{-H/2}^{H/2} = \frac{1}{12} MH^2
\end{aligned}$$

For the products, we can see that all integrals over z will vanish due to symmetry, so the only product of interest is:

$$\iiint -\rho xy dV = -\frac{1}{24} \rho HL^2B^2 = -\frac{1}{12} MLB.$$

This leads us to:

$$\mathbf{I} = \frac{1}{12} M \begin{pmatrix} 2L^2+H^2 & -LB & 0 \\ -LB & 2B^2+H^2 & 0 \\ 0 & 0 & 2B^2+2L^2 \end{pmatrix}.$$

(b) The z-axis is a principal axis, as for a rotation around this axis, that is for $\vec{\omega} = \omega_3 \hat{e}_3$, the angular momentum is parallel to the angular velocity. In this case, $\vec{L} = \mathbf{I}\vec{\omega} = \frac{1}{12} M(2B^2+L^2)\omega_3 \hat{e}_3 \parallel \vec{\omega}$.

Rotational Symmetry is not a requirement for the existence of principal axes. *Every* rigid body has three principal axes, so it is not surprising to find them in this example. Furthermore, there are symmetries related to $\hat{z}$, namely that the xy-plane is a (mirror)symmetry plane of the body.

[1] or, of course, $(0, B)$ for x and $(0, L(1-x/B))$ for y. Also, note that the order of integration is now important for x and y!

(c) The form of Newton's law we are looking for is $\vec{F} = \frac{d\vec{p}}{dt}$. For a rotation, this becomes $\vec{\Gamma} = \frac{d\vec{L}}{dt}$, with $\vec{\Gamma}$ the torque acting on the body, and $\frac{d\vec{L}}{dt}$ the change of the angular momentum *in an inertial system*. To derive the Euler equation(s), we need to transform this into the (non-inertial) body frame, where the time derivative gains an extra term:

$$\vec{\Gamma} = \dot{\vec{L}} + \vec{\omega} \times \vec{L},$$

where the "dot" now refers to a simple time derivative in the non-inertial coordinate system. As the the coordinate axes of the body frame are principal axes, we know that $\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$. Writing out the cross product, we obtain:

$$\begin{aligned}
\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2\omega_3 &= \Gamma_1 \\
\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 &= \Gamma_2 \\
\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1\omega_2 &= \Gamma_3
\end{aligned}$$

Without external torque and for $\lambda_1 = \lambda_2$ we get

$$\begin{aligned}
\lambda_1\dot{\omega}_1 - (\lambda_1 - \lambda_3)\,\omega_2\omega_3 &= 0 \\
\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 &= 0 \\
\lambda_3\dot{\omega}_3 &= 0
\end{aligned}$$

(d) We see that $\dot{\omega}_3 = 0$ and thus $\omega_3 = \text{const}$:

$$\begin{aligned}
\dot{\omega}_1 - \underbrace{\left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3\right)}_{\Omega_b}\omega_2 &= 0 \\
\dot{\omega}_2 - \left(\frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\right)\omega_1 &= 0
\end{aligned}$$

You can now either use the trick of turning the two equations into a single, complex differential equation by using $\omega_1 + i\omega_2 = \eta$ and therefore $\dot{\eta} = -i\Omega_b\eta$, or by directly trying the test solutions $\omega_1 = \omega_o \cos\Omega_b t$ and $\omega_2 = -\omega_o \sin\Omega_b t$.

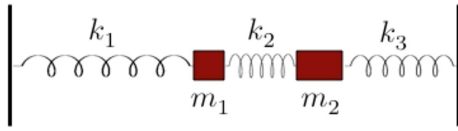
This describes a circular "wobble" of $\vec{\omega}$ around $\hat{e}_3$ with angular frequency $\Omega_b = \frac{1}{305}\omega_3$. As the latter is the rotation of earth (period roughly 1 day), the period of the Chandler Wobble is 305 days.

(e) The two assumptions that are violated are: (1) there are external (gravitational) torques acting on earth, and (2) Earth is *not* a rigid body (remember the tides?). *[NB:]* no points were given for pointing out that earth is not a perfect sphere (assumed by having more than one principal moment), suggesting different principal moments or axes (explicitly stated in question), or invoking general relativity.

6cm

3 Coupled Oscillations

Two blocks and three springs are connected as shown in the figure. All motion is horizontal. When the blocks are at rest, all springs are unstretched. **(total: 18 points)**



(a) Choose x_1 and x_2 as generalized coordinates for the displacement from the equilibrium positions of the blocks with masses m_1 and m_2, and find the equation of motion for both blocks. (5 points)

(b) Combine the set of of motions into matrix form $\mathbf{M}\ddot{\vec{x}} = -\mathbf{K}\vec{x}$. (3 points)

From now on, use $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$ and $k_3 = 2k$.

(c) Find the frequencies of (small) oscillations of the modes. (6 points)

(d) Find the normal modes of this system. Describe the physical motions to which the normal modes correspond. (4 points)

3.1 Solutions

(a) The equations of motion are found quite easily from Hooke's law $\vec{F} = -k\vec{x}$; as the springs are unstretched at rest:

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) &&= -(k_1 + k_2)x_1 + k_2x_2 \\ m_2\ddot{x}_2 &= -k_3x_2 - k_2(x_2 - x_1) &&= k_2x_1 - (k_2 + k_3)x_2 \end{aligned}$$

(b) The last simplification above directly gives us the matrix form:

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}$$

Note the sign convention!

(c) If you calculate the determinant ($\lambda = \frac{\omega^2}{\omega_o^2}$),

$$\det \begin{pmatrix} 5 - 2\lambda & -1 \\ -1 & 3 - \lambda \end{pmatrix} = 0 \quad \Rightarrow \quad 2\lambda^2 - 11\lambda + 14 = 0$$

Giving you the same frequencies (note that I changed to natural units, *i.e.* $\omega_o^2 = \frac{k}{m}$, which drops both k and m from the equation; λ is now a multiple of the natural frequency ω_o). For $\lambda_1 = 2$ we get

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \vec{a}_1 = 0 \quad \Rightarrow \quad \vec{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and for $\lambda_2 = 7/2$ we find:

$$\begin{pmatrix} -2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \cdot \vec{a}_2 = 0 \quad \Rightarrow \quad \vec{a}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

(d) This point and the next are closely connected, as we have to solve the eigenvalue and eigenvector problem. With the values for masses and spring constants, the equations of motion are

$$\begin{aligned} 2m\ddot{x}_1 &= -4kx_1 - k(x_1 - x_2) &&= -5kx_1 + kx_2 \\ m\ddot{x}_2 &= -2kx_2 - k(x_2 - x_1) &&= kx_1 - 3kx_2 \end{aligned}$$

and the matrices are now:

$$\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \cdot m \qquad \mathbf{K} = \begin{pmatrix} 5 & -1 \\ -1 & 3 \end{pmatrix} \cdot k$$

To state the eigenvalue problem, we write:

$$(\mathbf{K} - \omega^2 \mathbf{M})\vec{a} = 0$$

where ω is the eigenvalue (normal frequency) and $\vec{a}$ is an eigenvector (normal mode). You can attempt to solve this by finding the determinant $\det(\mathbf{K} - \omega^2\mathbf{M})$, and from there find the eigenvectors, or you can argue from physics and find the collective (centre of mass) motion and the relative motion of the two masses (generalized coordinates), that is the sum and difference of the two coupled equations of motion, and show that the equations *decouple*:

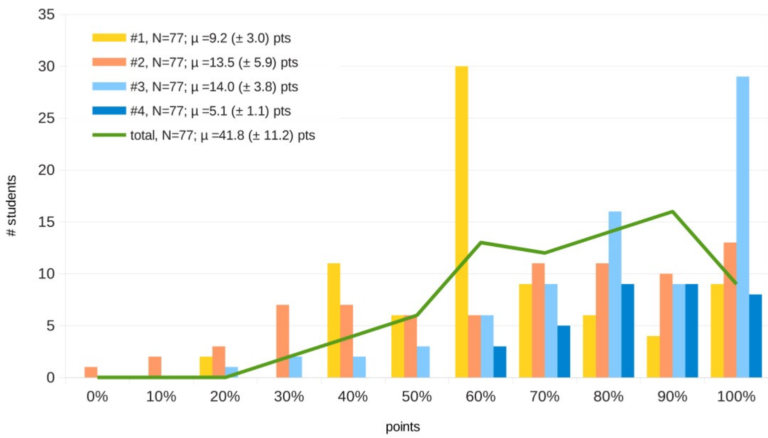
$$\begin{aligned} m(2\ddot{x}_1 + \ddot{x}_2) &= -2k(2x_1 + x_2) &&\Rightarrow \xi_1 = 2x_1 + x_2,\ \omega_1 = \sqrt{\frac{2k}{m}} = \sqrt{2}\omega_{\text{o}},\ \vec{a}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ 2m(\ddot{x}_1 - \ddot{x}_2) &= -7k(x_1 - x_2) &&\Rightarrow \xi_2 = x_1 - x_2,\ \omega_2 = \sqrt{\frac{7}{2}}\omega_{\text{o}},\ \vec{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

The one conceptual jump needed here is to recognize that if either of these coordinates disappears ($\xi_i(t) = 0$), you have found the amplitude $\vec{a}$ for the *other* normal mode! Alternatively you can also invert the definitions of ξ_1 and ξ_2 to find that $\xi_1 + \xi_2 = 3x_1$ and $\xi_1 - 2\xi_2 = 3x_2$.

Physically, you have thus shown that the normal modes are (1) a collective motion that does not stretch spring 2 ($x_1 = x_2$), and (1) an opposing motion where both masses oscillate against each other and the amplitude of m_2 is twice as big as that of m_1 ($x_1 = -2x_2$).

4 Multiple Choice

The final questions for this exam are multiple choice (on a separate sheet). Please make sure to remove the staple *cleanly* before you hand in the answer sheet. [total: 5+2 bonus points]



Cronbach Alpha	**0.80**	Cα should be between 0.6 and 0.8	Rit Item-test correlation
Overall Difficulty	**0.67**	should be above 0.1	Rir Item-rest correlation

	1a	**1b**	**1c**	**1d**	**2a**	**2b**	**2c**	**2d**	**2e**	**3a**	**3b**	**3c**	**3d**	**4a**	**1**	**2**	**3**	**4**
points	**6**	**5**	**2**	**2**	**8**	**4**	**3**	**4**	**3**	**5**	**3**	**6**	**4**	**7**	**15**	**22**	**18**	**7**
difficulty	0.58	0.45	0.87	0.84	0.66	0.63	0.79	0.42	0.54	0.88	0.91	0.83	0.48	0.73	0.61	0.61	0.78	0.73
Rit	0.60	0.49	0.41	0.38	0.67	0.74	0.57	0.76	0.57	0.63	0.36	0.64	0.68	0.74	0.69	0.88	0.79	0.74
Rir	0.50	0.40	0.36	0.32	0.51	0.67	0.50	0.69	0.49	0.56	0.30	0.57	0.58	0.69	0.51	0.59	0.59	0.69