Tentamen Voortgezette Mechanica

NS-350B, Blok 2, Midterm, 14 December 2017

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Tip: Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Double Pendulum

A mathematical pendulum of length l_1 and mass m_1 is attached in point O. A second pendulum of length l_2 and mass m_2 is attached at the end of the first pendulum. All connections move without friction and may be considered massless. [total: 15 points]

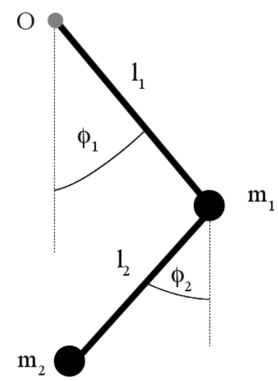


Figure 1: Double pendulum. Note that in the above situation, ϕ_2 is negative.

(a) In order to find the Lagrangian $\mathcal{L}(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2, t)$ that describes this system, find the kinetic energy T and potential energy U.

Show that our choice of ϕ_1 and ϕ_2 satisfies the requirements that a Lagrangian has for its generalized coordinates. Include a short explanation of the requirements. (5+1 points)

(b) Find the generalized force and generalized momentum of each coordinate, and use them to write down the equation of motion for ϕ_1 and ϕ_2. Work carefully – the expressions are complex and have multiple terms! (3+2 points)

(c) Consider the case that $m_1 \gg m_2$ and and $l_1 \sim l_2$; first, without calculation, describe what the motion of $\phi_1(t)$ should look like and then find the solution in the limit of small oscillations. (2+2 points)

1.1 Solution

(a) Concerning the generalized coordinates, remember that you need to be able to describe the positions of each mass in those coordinates (obviously true), that the coordinates are independent (they are), thus that they are are *holonomic*, *i.e.* that there is an equal number of coordinates and degrees of freedom (Two and two, respectively).

To find the kinetic and potential energy for the Lagrangian $\mathcal{L} = T - U$, we need to first calculate the position of the two masses as a function of ϕ_1, ϕ_2, and t:

$$\mathbf{x}_1 = \begin{pmatrix} l_1 \sin\phi_1 \\ -l_1 \cos\phi_1 \end{pmatrix} \tag{1}$$

$$\mathbf{x}_2 = \begin{pmatrix} l_1 \sin\phi_1 + l_2 \sin\phi_2 \\ -l_1 \cos\phi_1 - l_2 \cos\phi_2 \end{pmatrix} = \mathbf{x}_1 + l_2 \begin{pmatrix} \sin\phi_2 \\ -\cos\phi_2 \end{pmatrix} \tag{2}$$

For the potential energy we can directly use the y components $U = g(m_1 y_1 + m_2 y_2)$:

$$U = -g\left((m_1 + m_2)(l_1 \cos\phi_1) + m_2 l_2 \cos\phi_2\right), \tag{3}$$

while for the kinetic energy $T = \frac{1}{2}m_1\dot{\mathbf{x}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{x}}_2^2$ we need to first differentiate with respect to time:

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \begin{pmatrix} l_1\dot{\phi}_1\cos\phi_1 \\ l_1\dot{\phi}_1\sin\phi_1 \end{pmatrix} && (4)\\
\dot{\mathbf{x}}_2 &= \begin{pmatrix} l_1\dot{\phi}_1\cos\phi_1 + l_2\dot{\phi}_2\cos\phi_2 \\ l_1\dot{\phi}_1\sin\phi_1 + l_2\dot{\phi}_2\sin\phi_2 \end{pmatrix} = \dot{\mathbf{x}}_1 + l_2\dot{\phi}_2\begin{pmatrix}\cos\phi_2 \\ \sin\phi_2\end{pmatrix} && (5)
\end{aligned}$$

Collecting the terms, using the sum rules of sine an cosine, we find:

$$\begin{aligned}
\dot{\mathbf{x}}_1^2 &= l_1^2\dot{\phi}_1^2 && (6)\\
\dot{\mathbf{x}}_2^2 &= l_1^2\dot{\phi}_1^2 + 2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) + l_2^2\dot{\phi}_2^2 && (7)
\end{aligned}$$

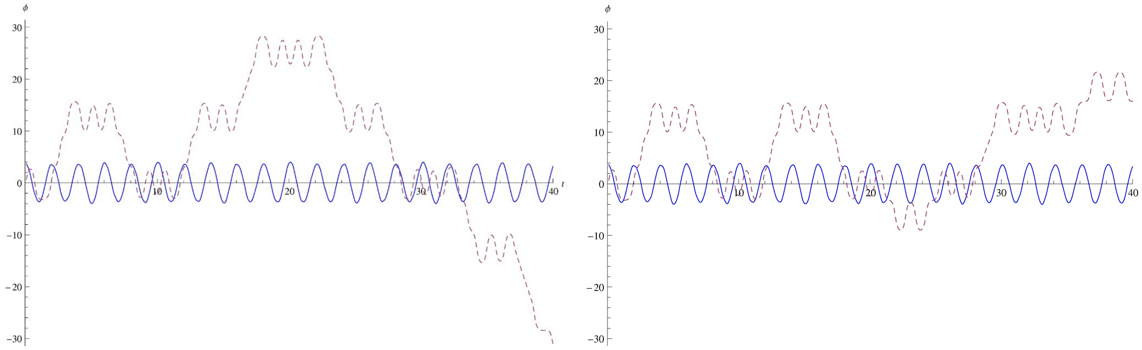
(b) To find the equation of motion for ϕ_1, we use the Euler-Lagrange-equation

$$\begin{aligned}
\overbrace{\frac{\partial\mathcal{L}}{\partial\phi_1}}^{\text{generalized force}} &= \frac{d}{dt}\overbrace{\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1}}^{\text{generalized moment}}\\
\frac{\partial\mathcal{L}}{\partial\phi_1} &= -m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - g(m_1+m_2)l_1\sin\phi_1 && (8)\\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} &= \frac{d}{dt}\Big((m_1+m_2)l_1^2\dot{\phi}_1 + m_2l_1l_2\dot{\phi}_2\cos(\phi_1-\phi_2)\Big)\\
&= (m_1+m_2)l_1^2\ddot{\phi}_1 + m_2l_1l_2\ddot{\phi}_2\cos(\phi_1-\phi_2) - m_2l_1l_2\dot{\phi}_2\sin(\phi_1-\phi_2)\Big(\dot{\phi}_1-\dot{\phi}_2\Big) && (9)\\
\ddot{\phi}_1 &= \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}\ddot{\phi}_2\cos(\phi_1-\phi_2) + \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}\dot{\phi}_2^2\sin(\phi_1-\phi_2) - \frac{g}{l_1}\sin\phi_1 && (10)\\
\frac{\partial\mathcal{L}}{\partial\phi_2} &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2}\\
\frac{\partial\mathcal{L}}{\partial\phi_2} &= m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - gm_2l_2\sin\phi_2 && (11)\\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} &= \frac{d}{dt}\Big(m_2l_2^2\dot{\phi}_2 + m_2l_1l_2\dot{\phi}_1\cos(\phi_1-\phi_2)\Big)\\
&= m_2l_2^2\ddot{\phi}_2 + m_2l_1l_2\ddot{\phi}_1\cos(\phi_1-\phi_2) - m_2l_1l_2\dot{\phi}_1\sin(\phi_1-\phi_2)\Big(\dot{\phi}_1-\dot{\phi}_2\Big) && (12)\\
\ddot{\phi}_2 &= -\frac{l_1}{l_2}\ddot{\phi}_1\cos(\phi_1-\phi_2) + \frac{l_1}{l_2}\dot{\phi}_1^2\sin(\phi_1-\phi_2) - \frac{g}{l_2}\sin\phi_2 && (13)
\end{aligned}$$

(c) If $m_1 \gg m_2$, and $l_1 \sim l_2$, the terms containing $\ddot{\phi}_2$ and $\dot{\phi}_2$ become negligible as $\frac{m_2}{m_1+m_2}\frac{l_2}{l_1} \to 0$, and we find that ϕ_1 moves as if there was no second pendulum:

$$\ddot{\phi}_1 = -\underbrace{\frac{g}{l_1}}_{\omega_0^2}\sin\phi_1.$$

This is actually visible in figure 2, which shows a very regular movement of m_1, while m_2 moves chaotically.



(a) $\phi_2(0) = -\pi/4$ rad, $m_1 > m_2$ (b) $\phi_2(0) = -\pi/4 - 0.00005$ rad, $m_1 > m_2$

Figure 2: ϕ_1 and ϕ_2 (dashed line) as a function of time; the initial values in (a) (b) are identical except for a small variation in $\phi_2(0)$.

The motion of the pendulum in this case (small oscillations) will be given by $\phi_1(t) = A\sin(\omega_o t + \delta)$. You can show this by either expanding the $\sin(\phi_1)$ term ($\phi_1 + \mathcal{O}(\phi_1^3)$) or by arguing that the potential U can be approximated by $\cos(\phi_1) = 1 - \frac{\phi_1^2}{2} + \mathcal{O}(\phi_1^4)$ and that the motion is thus harmonic (ignoring the trivial constant).

2 Projectile and Incline

A bullet of mass m is launched with initial speed v_o at an angle θ to the horizontal. You may neglect air resistance encountered by the projectile. The force of gravity acts vertically down ($a_g = -g\hat{y}$). Eventually the bullet lands on an incline which is at an angle α; upon impact, the projectile loses its velocity component normal to the incline and begins to slide up the incline. The friction encountered by the projectile on the incline is equal to μ times the normal force. [total: 10 points]

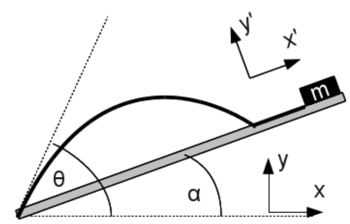


Figure 3: Mass on incline with friction.

(a) Determine all forces (magnitudes and directions) acting on the bullet before it lands on the incline and use Newton's laws to calculate how far up the incline (measured along the x' direction) the bullet lands. Find the velocity $\vec{v}_i$ of the bullet at the time of impact. *Hint:* as the bullet lands when $y' = 0$, consider using the coordinate system (x', y') for this calculation. (5 points)

(b) Determine all forces (magnitudes and directions) acting on the bullet after it lands on the incline and calculate how far up the incline (measured in the y direction) the bullet will travel from its point of impact. Check your result by comparing it to the case where there is no friction ($\mu = 0$, energy conservation). (4 points)

(c) How large does α have to be so that the bullet will slide back down the incline after it has come to a stop? (1 points)

2.1 Solution

(a) While in the air, the only force is that of gravity, $F_z = -mg\hat{\mathbf{y}} = -mg\cos\alpha\hat{\mathbf{y}}' - mg\sin\alpha\hat{\mathbf{x}}'$

$$\begin{aligned} v^o_{x'} &= v_o\cos(\theta-\alpha) \\ v^o_{y'} &= v_o\sin(\theta-\alpha) \\ a_{x'} &= -g\sin\alpha \\ a_{y'} &= -g\cos\alpha \end{aligned}$$

y' will reach its largest value (the apex in the x', y'-system) when $v_{y'} = 0$, that is to say at time

$$t_{\frac{1}{2}} = \frac{v_o\sin(\theta-\alpha)}{g\cos\alpha}$$

The mass will hit the incline at twice that, thus $t = 2t_{\frac{1}{2}}$. Furthermore we now that at that time:

$$\begin{aligned} v_{x'} &= v^o_{x'} + 2a_{x'}t_{\frac{1}{2}} = v_o\left(\cos(\theta-\alpha) - 2\tan\alpha\sin(\theta-\alpha)\right) \\ v_{y'} &= v^o_{y'} + 2a_{y'}t_{\frac{1}{2}} = -v^o_{y'} = -v_o\sin(\theta-\alpha) \\ x' &= v^o_{x'}t + \frac{1}{2}a_{x'}t^2 = \\ &= \frac{2v_o^2}{g\cos\alpha}\left(\sin(\theta-\alpha)\cos(\theta-\alpha) - \sin^2(\theta-\alpha)\tan\alpha\right) \text{ or} \\ &= \frac{2v_o^2}{g\cos\alpha}\left(\sin\theta\cos\theta - \cos^2\theta\tan\alpha\right), \end{aligned}$$

where the latter result comes from the calculation in which you did not follow the hint but used x and y as your coordinates straight away. The contact condition (when the bullet hits the incline) in this case is a bit more complicated ($y = x\tan\alpha$).

(b) After impact, there are three forces that act on the mass:

- gravity as before (downwards) $F_z = -mg\hat{\mathbf{y}} = -mg\cos\alpha\hat{\mathbf{y}}' - mg\sin\alpha\hat{\mathbf{x}}'$,
- the normal force opposing gravity (perpendicular to the incline) $F_N = mg\cos(\alpha)\hat{\mathbf{y}}'$, and
- friction (always opposite to the direction of motion along the incline) $F_w = -\mu F_N\hat{\mathbf{x}}' = -\mu mg\cos(\alpha)\hat{\mathbf{x}}'$.

The Kinetic Energy is given by:

$$\frac{1}{2}mv_{x'}^2 = \frac{1}{2}m\left(v_o\cos(\theta-\alpha) - 2v_o\tan\alpha\sin(\theta-\alpha)\right)^2$$

Conservation of energy tells us that this kinetic energy is conversed into potential energy and lost (well, conversed into heat) due to friction:

$$\frac{1}{2}m\left(v^o_{x'}\right)^2 = mgy + x'\mu mg\cos\alpha,$$

where $v^o_{x'}$ is the (x'-component of the) velocity at the moment of impact and x' en y both measured from the selfsame point. As we know that on the slope $y = x' \sin\alpha$ we finally find a semi-nice, short expression:

$$\frac{1}{2} m {v^o_{x'}}^2 = mgx' \left(\sin\alpha + \mu\cos\alpha\right) = mgy\left(1 + \frac{\mu}{\tan\alpha}\right),$$

and therefore again

$$\begin{aligned} x'_{\max} &= \frac{v_o^2}{2g\cos\alpha\left(\tan\alpha + \mu\right)} \left(\cos\left(\theta - \alpha\right) - 2\tan\alpha\sin\left(\theta - \alpha\right)\right)^2 \\ y_{\max} &= \frac{v_o^2}{2g\left(1 + \frac{\mu}{\tan\alpha}\right)} \left(\cos\left(\theta - \alpha\right) - 2\tan\alpha\sin\left(\theta - \alpha\right)\right)^2 \end{aligned}$$

If there is no friction, $\mu \to 0$, and we see that $y_{\max}(\mu = 0) = \frac{v_o^2}{2g\sin\alpha}\left(\cos\left(\theta - \alpha\right) - 2\tan\alpha\sin\left(\theta - \alpha\right)\right)^2$ which is exactly what we get from energy conservation $\frac{1}{2}mv_o^2 = mgy$, and the friction unsurprisingly stops the mass earlier.

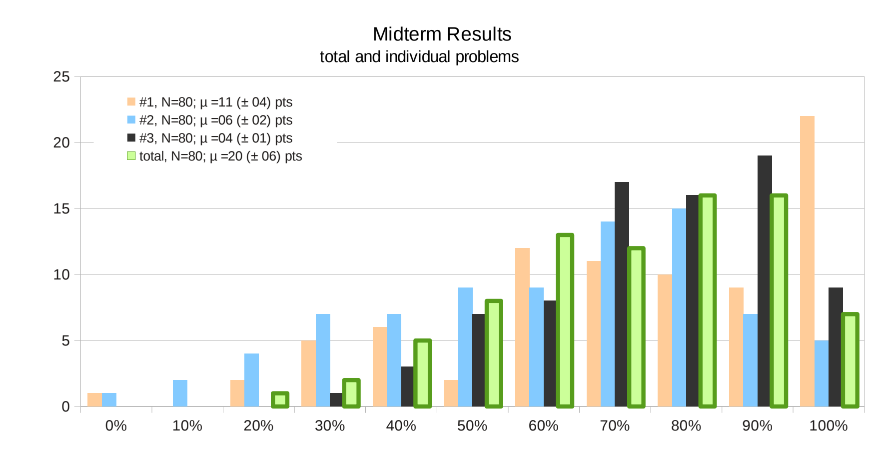
(c) Once the bullet has stopped, the component of gravity along the incline will accelerate it down the incline again *if* it is larger than the force of (static!) friction opposing it:

$$F_{\parallel} = mg\sin\alpha > \mu mg\cos\alpha = F_{\text{friction}},$$

which leads to $\tan\alpha > \mu$ (or $\alpha \geqslant \frac{\pi}{2}$ in case of $\mu \geqslant 1$, which is possible). For completeness sake you should also mention that the μ used so casually in this case is actually the static coefficient of friction μ_S which should be larger than the sliding or kinetic friction μ_K used before!

3 Multiple Choice

The final questions for this exam are multiple choice (on a separate sheet). Please make sure to remove the staple *cleanly* before you hand in the answer sheet. [total: 5 points]



Cronbach Alpha	**0.74**	Cα should be between 0.6 and 0.8
Overall Difficulty	**0.67**	should be above 0.1

Rit Item-test correlation
Rir Item-rest correlation

	1a	**1b**	**1c**	**2a**	**2b**	**2c**	**3**		**1**	**2**	**3**
points	6	5	4	5	4	1	5		15	10	5
difficulty	0.76	0.65	0.67	0.69	0.48	0.56	0.73		0.70	0.59	0.73
Rit	0.80	0.77	0.45	0.73	0.58	0.39	0.59		0.89	0.76	0.59
Rir	0.62	0.61	0.25	0.59	0.41	0.31	0.47		0.46	0.46	0.47