Tentamen Voortgezette Mechanica

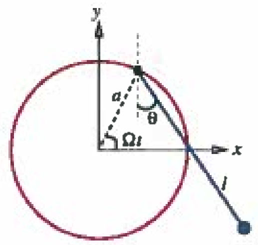
NS-350B, Blok 2, Final; January 31, 2019

Mark on *each* sheet clearly your **name** and **collegekaartnummer**.
Please use a **separate sheet** for each problem.
Tip: Read all questions and start with the one you find the easiest. Do not use too much time on any one question!

1 Rotating Pendulum

A rotating pendulum has the pivot point of a planar pendulum of length l and mass m forced to rotate with a constant angular velocity Ω on a circle of radius a. The origin of the coordinate system coincides with the center of the circle. This situation is shown in the figure. **(total: 18 points)**

Find:



(a) The Lagrangian $\mathcal{L}$ of the system. Why does it explicitly depend on time? (6+1 points)

(b) The Hamiltonian $\mathcal{H}$ of the system. (6 points)

(c) The Equations of Motion using the Hamilton equations. (5 points)

2 Non-Point-Masses: Rotation and Deformation

In this question, we will take a step away from point sources and deal with three dimensional bodies. Take a rectangular box of mass M and homogeneous density ρ. As shown in the figure on the right, the box has dimensions $h < b < l$ and is rotating around a point in the middle of one of its shortest edges. We will start by assuming that the body is *rigid*. **(total: 22 points)**

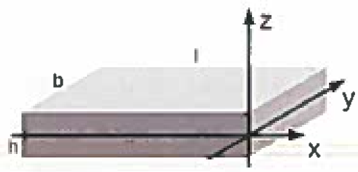


Figure 1: Rectangular Prism with homogeneous mass distribution.

(a) Calculate the inertia tensor $\mathbf{J}$ around the usual cartesian $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes, and the origin in the middle of the shortest edge of the box. (5 points)

(b) The inertia tensor consists of *moments of inertia* and *products of inertia*. Explain these two terms and give the physical meaning of the two elements of the inertial tensor J_{xy} and J_{yx} for a rotation $\vec{\omega}$ around an arbitrary axis. (3 points)

(c) Diagonalizing $\mathbf{J}$ is a messy affair and will teach us little, so for this part only you may assume that $l = b$: Find the principal moments and principal axes of this simpler inertia tensor $\mathbf{K} = \mathbf{J}_{l=b}$. (5 points)

(d) Using the parallel axis theorem on the original inertia tensor $\mathbf{J}$, find the inertia tensor $\mathbf{I}_{\mathrm{CM}}$ of the box through its center of mass. [*Hint:* $\mathbf{I}_{\mathrm{CM}}$ is diagonal in the given axes. If your calculation do not give you the correct result, explain why it must be diagonal for partial credit!] (2 points)

In reality, no body is completely rigid. We have found a generalized Hooke's law describing the relationship between the surface forces ("stress" tensor) and deformation in the form of the relevant parts of the derivatives matrix ("strain" tensor).

For the next section, $\mathbf{\Sigma} = \begin{bmatrix} xz & z^2 & 0 \\ z^2 & 0 & -y \\ 0 & -y & 0 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(e) Using the stress tensor $\mathbf{\Sigma}$, find the surface force on a small area dA of the surface $x^2+y^2+2z^2 = 4$ at the point $(1, 1, 1)$. Which type(s) of surface forces do you find? (3+1 points)

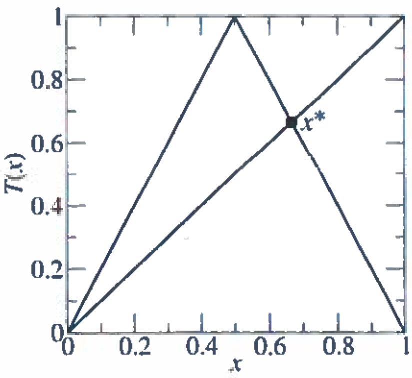
(f) Which two types of deformation does a general strain tensor contain, and where can you find them in the tensor? Which type of strain is given by the given tensor $\mathbf{E}$? (2+1 points)

3 The Tent Map

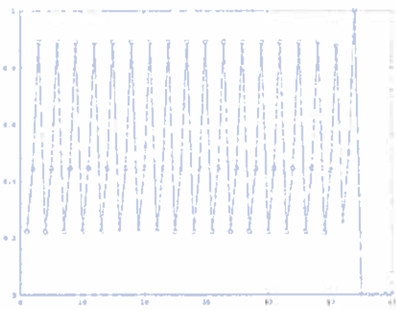
Consider the so-called tent map, the defintion of which is given by

$$T_\mu(x) := \mu \min(x, 1-x) = \begin{cases} \mu x & 0 \le x \le 1/2 \\ \mu(1-x) & 1/2 < x \le 1 \end{cases},$$

which maps a point x to $T_\mu(x)$. The graph for $T_2(x)$, that is to say the tent map in the chaotic regime with $\mu = 2$, simply looks like the triangle on the right (you can see why it is called a tent map). **(total: 15 points):**



(a) The tent map $T_2(x)$



(b) Iteration of T_2 starting at $\frac{2}{9}$

(a) Find $T_2^2(x) = T_2[T_2(x)]$ and plot it. (3 points)

(b) Calculate the fixed points x^* of $T_2(x)$ and $T_2^2(x)$. How many are there in each case? Are those fixed points stable? (4 points)

(c) What does the graph $T_2^n(x)$ look like and how many fixed points does it have (you do not have to calculate their values)? (3+1 points)

(d) In figure (b), you see the iteration for a three-cycle of T_2 starting at $x_o = 2/9$ (show that this starting value indeed results in a three-cycle!). Explain why the numerical calculation on a computer fails after the 50th iteration for this chaotic map. (4 points)

4 Multiple Choice

The final questions for this exam are multiple choice (on a separate sheet). Please make sure to remove the staple *cleanly* before you hand in the answer sheet. [total: 5 points]

Useful Formulas

$$\sin^2\alpha + \cos^2\alpha = -e^{i\pi} = 1, \qquad T_o \approx 2\pi\sqrt{\frac{\ell}{g}} \qquad \text{hoop: } I_{zz} = mr^2$$

$$c = \frac{\ell^2}{\gamma\mu} \quad E = \frac{\gamma^2\mu}{2\ell^2}\left(\epsilon^2 - 1\right) \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad \frac{c_1}{1 + \epsilon_1\cos(\phi + \delta_1)} = \frac{c_2}{1 + \epsilon_2\cos(\phi + \delta_2)}$$

$$U_{\text{cf}}(r) = \frac{\ell^2}{2\mu r^2} \qquad \mathcal{L}_{\text{rel}} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \qquad J_{ij} = I_{ij}^{\text{cm}} + m\left(\|\vec{a}\|^2\delta_{ij} - a_i a_j\right)$$

$$\mathbf{I} = \iiint dV\ \rho(x,y,z)\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

$$\mathcal{L} = T - V \qquad \frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}(q(t),\dot{q}(t),t)}{\partial\dot{q}} = 0$$

$$\mathcal{H} = \sum p_i\dot{q}_i - \mathcal{L} = T + U \qquad \dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i} \quad \dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$$

$$\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3) \quad \dot{\vec{L}} + \vec{\omega}\times\vec{L} = \vec{\Gamma} \qquad \frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \mathcal{O}(x^2)$$

$$\lambda\vec{\omega} = \mathbf{I}\vec{\omega} \qquad (\mathbf{I} - \lambda\mathbf{E_3})\vec{\omega} = 0 \qquad \det(\mathbf{I} - \lambda\mathbf{E_3}) = 0 \qquad \mathbf{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_i(d\vec{A}) = \sum_j \sigma_{ij}dA_j \qquad \vec{F}(d\vec{A}) = \mathbf{\Sigma}d\vec{A} = \mathbf{\Sigma}\hat{n}dA \qquad \mathbf{\Sigma} = \alpha e\mathbf{1} + \beta\mathbf{E}'$$

For a surface and a point on that surface: $S(x,y,z) = 0,\ P \in S:\quad \vec{n} = \nabla S$

$\int_0^{2\pi}\sin^n(x)\cos^m(x)dx = 0$ for odd m or n. $\qquad \int_0^{2\pi}\cos^2(x)dx = \int_0^{2\pi}\sin^2(x)\,dx = \pi$

$$\arctan 0 = 0 \qquad \arctan\pm\infty = \pm\tfrac{1}{2}\pi \qquad \det(\mathbf{K} - \omega^2\mathbf{M}) = 0$$

$$f(x^*) = x^* \qquad f(x_a) = x_b \wedge f(x_b) = x_a \Rightarrow f(f(x^*)) = x^* \qquad x_t = x^* + \epsilon \Rightarrow \epsilon_{t+1} \approx f'(x^*)\epsilon_t$$

$$f^{(n)}(x) = \underbrace{f(f(\cdots f(x)\cdots))}_{\text{ntimes}}$$

Voortgezette Mechanica (NS-350B) **Final Exam 2018/19; Multiple-Choice Part**

No documents allowed. The use of electronic devices is forbidden.
Questions using the sign ♣ may have zero, one or several correct answers and are worth 3 points.
All other questions have a single correct answer and are worth 2 points.

Question 1 ♣ Which statements about the stress and strain tensors are correct?

(A) Stress and Strain tensor are related by the generalized Hooke's law.

(B) The Strain tensor is composed of a dilatory part on the diagonal, and a deviatoric part (representing shear strain) on the off-diagonal.

(C) Volume forces, such as gravity, occur on the diagonal of the stress tensor.

(D) The Strain tensor **E** contains only the antisymmetric part of the derivatives matrix.

(E) There are three independent moduli describing an elastic body, namely Young's, Bulk, and Shear Modulus.

Question 2 ♣ In light of special relativity, which of the following statements are true?

(A) Events that occur simultaneously in one inertial frame also occur simultaneously in all other inertial frames.

(B) The speed of light in vacuum has the same value c in every direction in all inertial frames.

(C) The contracted length $l = \frac{l'}{\gamma}$ of a moving object only depends on the magnitude, not the direction of its movement.

(D) The relativistic velocity-addition formula only affects the velocity component parallel to the relative motion.

(E) If S is an inertial frame, any frame S' moving with a constant velocity relative to S is also an inertial frame

Question 3 Calculating a Hamiltonian, one has to follow 6 steps. Chose the right/most logical order:

A Write out the Hamilton's equations.

B Find the generalized momenta p.

C Solve for the $\dot{q}$ in terms of q and p.

D Choose suitable generalized coordinates.

E Write down the kinetic and potential energy.

F Write down the Hamiltonian $\mathcal{H} = T + U$, or when in doubt, using its formal definition.

(A) A – F – E – B – C – D

(B) D – E – B – C – F – A

(C) E – F – D – B – C – A

(D) D – A – B – E – C — F

(E) A – B – C – E – D –F

Question 4 ♣ Which of the following statements apply to the Inertia Tensor of an arbitrary rigid body?

(A) All elements of an Inertia Tensor I_{ij} have to be positive.

(B) Every Inertia Tensor has three distinct eigenvalues or *principal moments* which are different from each other.

(C) The Inertia Tensor is a function of the shape of the body only and does not depend on the choice of origin or axes.

(D) Every Inertia Tensor can be diagonalized.

(E) An axis of rotational symmetry is always a principal axis of a body.

Question 5 The change from $\mathcal{L}$ to $\mathcal{H}$ is an example of a

(A) Laplace Transformation (B) Fourier Transformation
(C) Lagrange Transformation (D) La Tène Transformation
(E) Legendre Transformation

Question 6 The Euler Angles θ, ϕ, and ψ allow us to connect the Euler equation in the body frame to an inertial frame. It turns out that θ is moving in an effective potential, which causes a nodding motion called ...

(A) nutation. (B) free rotation. (C) steady precession.
(D) Chandler wobble. (E) free precession.

Question 7 ♣ Coupled oscillations can be described in terms of *normal modes*. Which of these statements concerning normal modes are true?

(A) In a symmetric case (same spring constant and masses), all n normal modes of a coupled oscillator have the same ω.

(B) Any periodic motion of a coupled oscillator can be written as a (weighted) sum of normal modes.

(C) Every coupled oscillator has exactly one normal mode.

(D) A normal mode is any motion in which all n coordinates oscillate sinusoidally with the same frequency ω.

(E) Any periodic motion of a coupled oscillator has to be exactly one of the normal modes.

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Question 8 A question about the order of magnitude of time dilation. Suppose a certain event happens at intervals of exactly one hour in an inertial frame moving at a steady speed of 300 m s^{-1}. When measured from an inertial frame at rest, the interval will appear ...

(A) ...shorter by half a nanosecond.

(B) ...exactly the same length in time, as both are inertial frames.

(C) ...longer by half a millisecond.

(D) ...shorter by 2 milliseconds.

(E) ...longer by 2 nanoseconds.