MID-EXAM ADVANCED MECHANICS - with answers
12 DECEMBER 2019, 13:30-15:30 hours

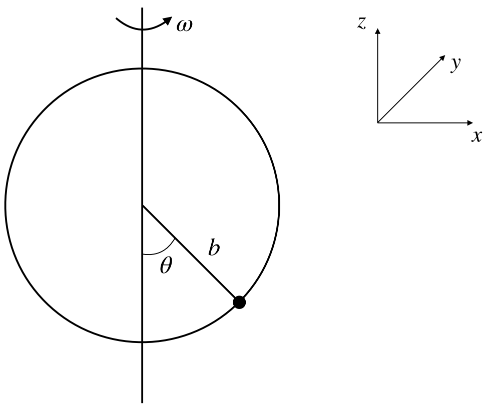
Three problems (all items have a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets.

Problem 1

A point mass m is threaded on a frictionless circular wire hoop of radius b. The hoop lies in a vertical plane, which rotates about the hoop's vertical diameter with a constant angular velocity ω. The position of the point mass is specified by the angle θ measured up from the vertical (see figure).



a. Draw all the forces (physical and inertial) acting on the point in a reference frame rotating with the hoop. Be clear about the names and directions of these forces.

b. Show that the equations of motion (in the same reference frame) are, in polar coordinates,

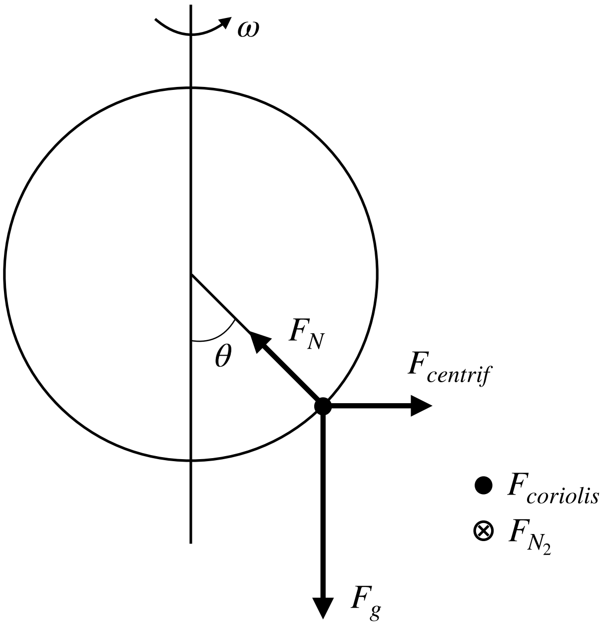
$$\ddot{r} = 0$$
$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{b}\right) \sin\theta$$

c. From now on we are going to consider only the second equation of motion, as there is no motion in the radial direction. It is easy to prove that $\theta^* = 0$ is an equilibrium position for the point mass. Find the other equilibria (in case they exist), or demonstrate that there are no other equilibria.

d. Consider an initial condition $\theta_0 << 1$ and $\dot{\theta}(t = 0) = 0$. Derive the solution of the equation of motion.

Solution

a. The forces acting on the point mass, assuming that it is moving anti-clockwise, are drawn in the figure below. Note that $\mathbf{F}_{Coriolis}$ and $\mathbf{F}_{N_2}$, respectively pointing outside and inside the plane xz (along $-y$ and y), are also applied to the point mass. $\mathbf{F}_N$ and $\mathbf{F}_{N_2}$ represent the normal forces exerted by the hoop. The centripetal force is not an actual force, but rather the resultant force between the radial components of $\mathbf{F}_g$ and $\mathbf{F}_{centrif}$ and $\mathbf{F}_N$. As we will see in the next item, such force is not zero, as long as the point is moving along the hoop.



b. The problem can be solved in polar or cartesian coordinates. In both cases, we will make use of equation FC 5.3.2 (be careful with the signs):

$$m\mathbf{a'} = \mathbf{F} - m\mathbf{A}_0 + \mathbf{F}_{Coriolis} + \mathbf{F}_{trans} + \mathbf{F}_{centrif} \tag{1}$$

where the resultant of the physical forces is given by

$$\mathbf{F} = \mathbf{F}_g + \mathbf{F}_N + \mathbf{F}_{N_2}$$

and the inertial forces

$$\mathbf{F}_{Coriolis} = -2m\boldsymbol{\omega}\times\mathbf{v'}$$

$$\mathbf{F}_{trans} = -m\dot{\boldsymbol{\omega}}\times\mathbf{r'} = 0 \qquad \text{as} \quad \boldsymbol{\omega} = \text{const.}$$

$$\mathbf{F}_{centrif} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')$$

$m\mathbf{A}_0 = \mathbf{0}$ as the reference frame is not trasnslating respect to the inertial system.

Remember that $\boldsymbol{\omega}$ is the angular velocity of the rotating system (the hoop) and $\mathbf{r}'$ and $\mathbf{v}'$ represent, respectively, position and velocity of the point in the rotating reference frame. Notice that $\mathbf{v}' \neq \mathbf{0}$, as the point is moving respect to the hoop (in the most general case). Since the point is not moving in the direction perpendicular to the hoop

$$\mathbf{F}_{Coriolis} = -\mathbf{F}_{N_2} \tag{2}$$

Polar coordinates

The plane in which in motion takes place is xz. Therefore, we can solve the equation of motion in this plane and consider a coordinate system

$$\mathbf{r}' = \begin{pmatrix} r \\ \theta \end{pmatrix}$$

where we dropped the primes to simplify the notation. With this choice of coordinates, the acceleration can be written as (see equation sheet)

$$\mathbf{a}' = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{pmatrix}$$

As we know that the point moves along the hoop, r will stay constant throughout the motion, in particular $r = b$. Therefore

$$\ddot{r} = \dot{r} = 0$$

which is our first result. Hence, the acceleration can be rewritten as

$$\mathbf{a}' = \begin{pmatrix} -b\dot{\theta}^2 \\ b\ddot{\theta} \end{pmatrix} \tag{3}$$

In order to write down the equations of motion we need to express the forces in radial components and in terms of b and θ.

$$|\mathbf{F}_{centrif}| = |-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')| = m\omega^2 b\sin(\pi-\theta) = m\omega^2 b\sin\theta$$
$$|\mathbf{F}_g| = mg$$
$$|\mathbf{F}_N| = F_N$$

If we project the three forces on the directions of r and θ, we obtain

$$\begin{aligned} F^{(r)}_{centrif} &= m\omega^2 b\sin^2\theta & \qquad F^{(\theta)}_{centrif} &= m\omega^2 b\sin\theta\cos\theta \\ F^{(r)}_g &= mg\cos\theta & F^{(\theta)}_g &= -mg\sin\theta \\ F^{(r)}_N &= -F_N & F^{(\theta)}_N &= 0 \end{aligned}$$

We can insert the equations above, together with equation (3), in the vectorial equation of motion (1) and obtain

$$\begin{aligned} -mb\dot{\theta}^2 &= m\omega^2 b\sin^2\theta + mg\cos\theta - F_N \\ mb\ddot{\theta} &= m\omega^2 b\sin\theta\cos\theta - mg\sin\theta \end{aligned}$$

From the second equation we can obtain the desired second result

$$\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{b}\right)\sin\theta \tag{4}$$

Notice the the first equation indicates that the force along the r-direction is generally not zero. The result force is indeed necessary for the point to move along the hoop (centripetal force).

Cartesian coordinates

The position vector of the system can be written as

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} b\sin\theta \\ 0 \\ -b\cos\theta \end{pmatrix}$$

where, once more, we dropped the primes. Notice that, since the reference frame is rotating with the hoop, the point will not move along the $y-$direction. Indeed, the forces along this direction balance out (see equation (2)). Again, we need to find the acceleration vector and then the forces expressed in cartesian coordinates. We have

$$\dot{\mathbf{r}}' = \begin{pmatrix} b\dot{\theta}\cos\theta \\ 0 \\ b\dot{\theta}\sin\theta \end{pmatrix}$$

and

$$\ddot{\mathbf{r}}' = \begin{pmatrix} b\ddot{\theta}\cos\theta - b\dot{\theta}^2\sin\theta \\ 0 \\ b\ddot{\theta}\sin\theta + b\dot{\theta}^2\cos\theta \end{pmatrix} \tag{5}$$

The forces read

$$\mathbf{F}_{centrif} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') = -m\boldsymbol{\omega}\times\begin{pmatrix} 0 \\ \omega b\sin\theta \\ 0 \end{pmatrix} = -m\begin{pmatrix} -\omega^2 b\sin\theta \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{F}_g = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

$$\mathbf{F}_N = \begin{pmatrix} -F_N\sin\theta \\ 0 \\ F_N\cos\theta \end{pmatrix}$$

We can insert the equations above, together with equation (5), in the vectorial equation of motion (1) and obtain (for the motion along x and z)

$$mb\ddot{\theta}\cos\theta - mb\dot{\theta}^2\sin\theta = m\omega^2 b\sin\theta - F_N\sin\theta$$
$$mb\ddot{\theta}\sin\theta + mb\dot{\theta}^2\cos\theta = -mg + F_N\cos\theta$$

If we multiply the first equation by $\cos\theta$ and the second one by $\sin\theta$ and then sum them, we obtain

$$mb\ddot{\theta} = m\omega^2 b\sin\theta\cos\theta - mg\sin\theta,$$

which again brings to the desired result.

c. An equilibrium point is any value of θ (let's call it θ_0) satisfying the following condition: if the point is placed at rest ($\dot{\theta}(t=0)=0$) at $\theta(t=0)=\theta_0$, then it will remain at rest at θ_0 for all times. This condition is guaranteed if $\ddot{\theta}=0$, as, in this case, the rest condition does not change. Therefore, in order to find the equilibria of the system, we need to impose

$$\left(\omega^2\cos\theta-\frac{g}{b}\right)\sin\theta=0$$

The equation above is satisfied if

$$\sin\theta=0$$

which, in the interval $\theta\in[0,2\pi)$, has $\theta_1=0$ and $\theta_2=\pi$ as solutions. Other solutions can be found by solving

$$\left(\omega^2\cos\theta-\frac{g}{b}\right)=0 \qquad\Rightarrow\qquad \cos\theta=\frac{g}{\omega^2 b}$$

The last equation has solutions as long as $\frac{g}{\omega^2 b}\leq 1$ and they can be written as

$$\theta_{3,4}=\pm\arccos\left(\frac{g}{\omega^2 b}\right)$$

To summarise, the equilibria are

$$\begin{aligned}\theta_1=0,\quad \theta_2=\pi \qquad &\text{if}\qquad \omega^2<g/b\\ \theta_1=0,\quad \theta_2=\pi,\quad \theta_{3,4}=\pm\arccos\left(\frac{g}{\omega^2 b}\right)\qquad &\text{if}\qquad \omega^2\geq g/b\end{aligned}$$

d. When $\theta_0<<1$, we can write

$$\theta=0+\theta'$$

where θ' is small. Therefore, we can expand (4) with the Taylor series of the trigonometric functions and obtain, at the first order,

$$\ddot{\theta}'=\left(\omega^2-\frac{g}{b}\right)\theta'$$

To be able to solve this equation, we need to consider three different cases

1. $\omega^2-\frac{g}{b}=-\alpha^2<0$

   The equation of motion can be written as

   $$\ddot{\theta}'=-\alpha^2\theta'$$

   and its general solution reads

   $$\theta'(t)=A\cos(\alpha t)+B\sin(\alpha t)$$

   where A and B are constants to be determined using the initial conditions $\theta'(0)=\theta_0$ and $\dot{\theta}'(0)=0$. The final solution is

   $$\theta'(t)=\theta_0\cos(\alpha t)$$

   The equilibrium is therefore stable and the point will oscillate (with a small amplitude θ_0) around it.

2. $\omega^2 - \frac{g}{b} = \alpha^2 > 0$
   The equation of motion can be written as

   $$\ddot{\theta}' = \alpha^2 \theta'$$

   and its general solution reads

   $$\theta'(t) = A \exp(\alpha t) + B \exp(-\alpha t)$$

   The final solution is

   $$\theta'(t) = \frac{\theta_0}{2} [\exp(\alpha t) + \exp(-\alpha t)] = \theta_0 \cosh(\alpha t)$$

   The equilibrium is unstable in this case: the point will accelerate away from the bottom. Note that this solution is valid only for small angles.

3. $\omega^2 - \frac{g}{b} = \alpha^2 = 0$
   In this case

   $$\ddot{\theta}' = 0$$

   The general solution is

   $$\theta'(t) = A + Bt$$

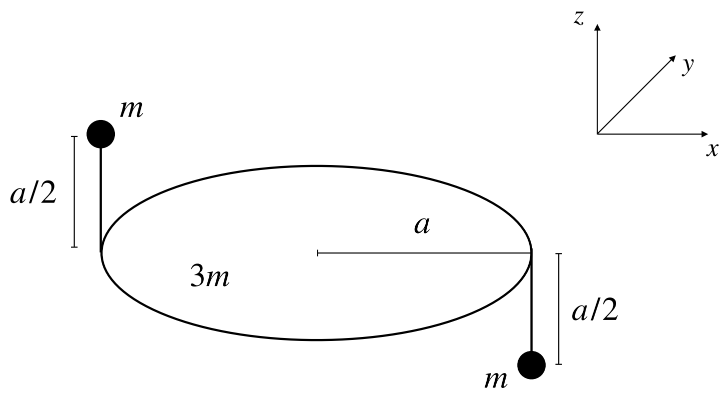
   and the final result is

   $$\theta'(t) = \theta_0$$

   The point will simply stay at θ_0.

Problem 2

Two point masses m are, by means of rigid massless rods, connected to the edge of a flat disk (radius a and mass $3m$, homogeneous mass distribution ρ). The length of each rod is $a/2$ (see situation sketch). Choose the x- and y-axis in the plane of the disk, the z-axis perpendicular to the disk and take as the origin the center of mass of the disk.



a. Demonstrate that the moment of inertia tensor of this object, in the given coordinate system, can be written as

$$\begin{pmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{pmatrix}$$

and express the components I_{xx}, I_{yy}, I_{zz} and I_{xz} in terms of a and m.

b. Calculate the angular momentum vector of the object in the case that it rotates about the z-axis with angular velocity ω.

c. Calculate the torque that is required to maintain the rotation about the z-axis.

d. Find the principal axes of rotation of the object, as well as the corresponding principal moments of inertia. Explain the physical meaning of principal axes.

Mid-term Advanced Mech 12/12/19 , problem 2

a. $\cdot I_{xy} = I_{yx} = 0$, $I_{yz} = I_{zy} = 0$ } because the point masses are in the x–z plane and the disk has z-axis as symmetry axis, so its moment of inertia tensor is already on diagonal form.

Alternative: do the computations

$\cdot\ I_{xx} = \sum_{i=1}^{2} m_i (y_i^2 + z_i^2) + \iint \rho (y^2+z^2)\, dx\, dy$

$= 2m\left(\frac{a}{2}\right)^2 + \int_0^{2\pi}\int_0^{a} \rho\, r^2 \cos^2\theta\, r\, dr\, d\theta = \frac{1}{2} ma^2 + \frac{3}{4} ma^2 = \frac{5}{4} ma^2$

• Used: coordinates of mass points: $(a, 0, -\frac{1}{2}a)$ and $(-a, 0, \frac{1}{2}a)$ and mass of disk $3m = \rho \pi a^2$

• $I_{yy} = 2m \frac{5}{4} a^2 + \frac{3}{4} m a^2 = \frac{13}{4} ma^2$

(used: $I_{yy}(\text{disk}) = I_{xx}(\text{disk})$ because of symmetry

• $I_{zz} = 2ma^2 + \frac{3}{2} m a^2 = \frac{7}{2} ma^2$

(used: $I_{zz}(\text{disk}) = I_{xx}(\text{disk}) + I_{yy}(\text{disk})$: perpendicular axis theorem)

• $I_{xz} = I_{zx} = -2ma(-\frac{1}{2}a) + 0 = ma^2$

In matrix form: $\underline{\underline{I}} = \frac{1}{4} ma^2 \begin{pmatrix} 5 & 0 & 4 \\ 0 & 13 & 0 \\ 4 & 0 & 14 \end{pmatrix}$

b. $\vec{L} = \underline{\underline{I}} \cdot \vec{\omega}$; here $\vec{\omega} = (0, 0, \omega)$

So

$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \frac{1}{4} ma^2 \begin{pmatrix} 5 & 0 & 4 \\ 0 & 13 & 0 \\ 4 & 0 & 14 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \frac{1}{4} ma^2 \omega \begin{pmatrix} 4 \\ 0 \\ 14 \end{pmatrix}$

c. Angular momentum balance in co-rotating frame:

$\left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega} \times \vec{L} = \vec{N}$

In this frame $\left(\frac{d\vec{L}}{dt}\right)_{rot} = 0$, since rotation is fixed : $\dot{\vec{\omega}} = 0$

So

$\vec{N} = \vec{\omega} \times \vec{L} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ L_1 & L_2 & L_3 \end{vmatrix} = -\vec{i}\,\omega L_2 + \vec{j}\,\omega L_1$

$= \vec{j}\, \frac{4}{4} ma^2 \omega^2$

(pto)

d. Solve eigenvalue problem :

$$\det(\underline{\underline{I}} - \lambda \underline{\underline{1}}) = 0$$

Set $\lambda = \frac{1}{4} m a^2 \mu$

$$\rightarrow (5-\mu)\left[(13-\mu)(14-\mu)\right] - 4\left[4(13-\mu)\right] = 0$$

$$(13-\mu)\left[\mu^2 - 19\mu + 54\right] = 0 \quad \rightarrow \quad \mu = \frac{19 \pm \sqrt{145}}{2}, \quad \mu = 13$$

$\mu = 13 \leftarrow \lambda = \frac{13}{4} m a^2$: corresponding eigenvector $(0,1,0)$

so principal axis : y-axis

$\mu = \mu_\pm \rightarrow \lambda_\pm = \frac{1}{4} m a^2 \mu_\pm$:

$$\begin{pmatrix} 5-\mu_\pm & 0 & 4 \\ 0 & 13-\mu_\pm & 0 \\ 4 & 0 & 14-\mu_\pm \end{pmatrix} \begin{pmatrix} e_{1\pm} \\ e_{2\pm} \\ e_{3\pm} \end{pmatrix} = 0$$

$$\rightarrow e_{2\pm} = 0, \quad \underbrace{e_{1\pm} = 1, \quad e_{3\pm} = -\tfrac{1}{4}(5-\mu_\pm)\, e_{1\pm}}$$

unnormalised eigenvector
divide by its length to make it a vector of length 1.

Meaning of principal axes : along these axes a torque-free, fixed rotation can be maintained.

Problem 3

Consider a central force $\mathbf{F} = f(r)\frac{\mathbf{r}}{r}$ and velocity vector $\mathbf{v}$ in $\mathbb{R}^3$.

a. Find the components of the antisymmetric part of dyad $\mathbf{F}\mathbf{v}$.

b. Calculate $\varepsilon_3 \vdots \varepsilon_3$,
i.e., the three-fold contraction of the ε_3 tensor with itself.
Explain how you obtain your answer.

c. Calculate Grad $\mathbf{F}$.

Midterm Adv. Mech 12/12/19 problem 3

a) $\vec{F}\vec{v}$ is a tensor of rank 2 with components $\{\vec{F}\vec{v}\}_{ij} = F_i v_j$

Use $F_i = f(r)\frac{x_i}{r}$.

Now the antisymmetric part of a second rank tensor T is

$$T^A = \frac{1}{2}(T - \tilde{T})$$

So the antisymmetric part of $\vec{F}\vec{v}$ is

$$\vec{F}\vec{v}^{\,A} = \frac{1}{2}(\vec{F}\vec{v} - \widetilde{\vec{F}\vec{v}}) = \frac{1}{2}(\vec{F}\vec{v} - \vec{v}\vec{F})$$

which has components $\boxed{\{\vec{F}\vec{v}^{\,A}\}_{ij} = \frac{1}{2}\frac{f(r)}{r}(x_i v_j - x_j v_i)}$

b) $\mathcal{E}_3 : \mathcal{E}_3$ is a tensor of rank $3+3-3\cdot 2 = 0$,

so it is a scalar:

$$\mathcal{E}_3 : \mathcal{E}_3 = \varepsilon_{ijk}\varepsilon_{ijk}$$

This is the sum of each component of $\mathcal{E}_3$ multiplied by itself.

Now according to the definition of $\mathcal{E}_3$:

$\varepsilon_{ijk}\varepsilon_{ijk}$ for fixed i,j,k is 1 if $\{i,j,k\}$ any permutation of $\{1,2,3\}$ and is 0 in all other cases.

The number of permutations is $3! = 6$

So $\boxed{\mathcal{E}_3 : \mathcal{E}_3 = 6.}$

c) Grad $\vec{F}$ is a tensor of rank 2 with components

$$\{\text{Grad}\,\vec{F}\}_{ij} = \nabla_i F_j = \frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left[f(r)\frac{x_j}{r}\right]$$

$$= \frac{\partial f}{\partial x_i}\frac{x_j}{r} + \frac{f(r)}{r}\underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}} + f(r)\,x_j\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right)$$

Use $\frac{\partial f}{\partial x_i} = \frac{df}{dr}\frac{\partial r}{\partial x_i} = \frac{df}{dr}\frac{x_i}{r}$ since $r = (x_k^2)^{1/2}$

$$\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x_i} = \frac{-x_i}{r^3}$$

So $$\{\text{Grad}\,\vec{F}\}_{ij} = \frac{f}{r}\delta_{ij} + \frac{1}{r^2}\left(\frac{df}{dr} - \frac{f}{r}\right)x_i x_j$$

$\rightarrow$ $\boxed{\text{Grad}\,\vec{F} = \frac{f}{r}\underline{\underline{1}} + \frac{1}{r^2}\left(\frac{df}{dr} - \frac{f}{r}\right)\vec{r}\vec{r}}$