SAMPLE FINAL EXAM ADVANCED MECHANICS, January 2020, time: 2 hours

Three problems (all items have a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets and add your name on them.

Problem 1

Three point masses m_1, m_2 and m_3 move in a three-dimensional space under influence of only gravitational forces that they exert on each other. The gravitational potential energy due to two point masses i and j is given by

$$V_{ij} = \frac{-Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|},$$

where G is the universal gravitational constant and $\mathbf{r}_i$ the position vector of point mass m_i. Use as generalised coordinates the cartesian coordinates (x_i, y_i, z_i) of each mass point with respect to a fixed origin.

a. Find the Hamiltonian function H for this system.

b. Derive the Hamiltonian canonical equations for coordinate x_1 and its associated conjugate momentum $p_{1,x}$, where $p_{1,x}$ is the x-component of $\mathbf{p}_1$.

   (If you do not have the answer of item a, use

$$H = \alpha\, p_{1,x}^2 + \beta\, p_{1,x} + \frac{\gamma}{\left((x_1 - \hat{x})^2 + \rho^2\right)^{1/2}},$$

   where $\alpha, \beta, \gamma, \hat{x}$ and ρ are constants.)

c. How many of Hamilton's canonical equations of this system are independent? Explain your answer.

See next page for problem 2

Solution

a. Use definition of H on equation sheet, or argue that $H = T + V$ (coordinates do not depend on time).
Thus,

$$T = \sum_{i=1}^{3} \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\,,$$

$$V = \frac{-Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{-Gm_1m_3}{|\mathbf{r}_1 - \mathbf{r}_3|} + \frac{-Gm_2m_3}{|\mathbf{r}_2 - \mathbf{r}_3|}\,.$$

The Hamiltonian function is to be expressed in terms of generalised coordinates and generalised momenta. Here,

$$\begin{aligned} p_{i,x} &\equiv \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i\,, && i = 1,2,3\,, \\ p_{i,y} &\equiv \frac{\partial L}{\partial \dot{y}_i} = \frac{\partial T}{\partial \dot{y}_i} = m_i \dot{y}_i\,, && i = 1,2,3\,, \\ p_{i,z} &\equiv \frac{\partial L}{\partial \dot{z}_i} = \frac{\partial T}{\partial \dot{z}_i} = m_i \dot{z}_i\,, && i = 1,2,3\,. \end{aligned}$$

Note: it has been used that $L = T - V$ and V does not depend on the velocities.

Use these relations to express T in terms of $p_{i,x}, p_{i,y}, p_{i,z}$. This finally gives

$$H = T + V = \sum_{i=1}^{3} \frac{(p_{i,x}^2 + p_{i,y}^2 + p_{i,z}^2)}{2m_i} - \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{Gm_1m_3}{|\mathbf{r}_1 - \mathbf{r}_3|} - \frac{Gm_2m_3}{|\mathbf{r}_2 - \mathbf{r}_3|}\,.$$

b. Use information from equation sheet and result of item a to find

$$\dot{x}_1 \equiv \frac{\partial H}{\partial p_{1,x}} = \frac{p_{1,x}}{m_1}$$

and

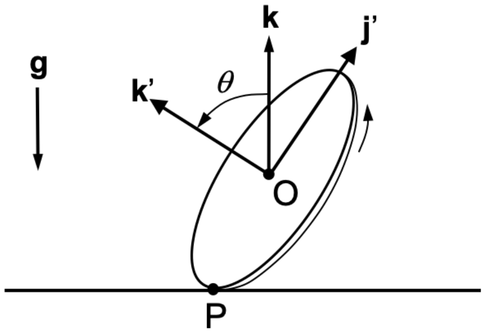
$$\dot{p}_{1,x} \equiv -\frac{\partial H}{\partial x_1} = \frac{Gm_1m_2(x_1 - x_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} + \frac{Gm_1m_3(x_1 - x_3)}{|\mathbf{r}_1 - \mathbf{r}_3|^2}\,.$$

c. This system has a total of 18 equations (for $x_i, y_i, z_i, p_{i,x}, p_{i,y}, p_{i,z}$ and $i = 1, 2, 3$). However, 11 of them are independent, because there are seven conserved quantities:
- total energy (H does not depend explicitly on time;
- total linear momentum in the x, y and z directions, because there are no external forces;
- total angular momentum in the x, y and z directions, as there are no external torques.

Problem 2

A coin is steadily rolling on a perfectly rough surface (see figure). The coin is a thin circular disk with radius a, mass m, moment of inertia I with respect to axes in the plane of the coin and moment of inertia I_s along its symmetry axis.

The velocity of the centre of mass of the coin is $\mathbf{v}_{cm}$ and its angular velocity is $\boldsymbol{\omega}$. The contact point between the coin and the surface is denoted by P and the origin O is at the centre of mass of the coin. Unit vector $\mathbf{j}'$ is in the direction from P to O, unit vector $\mathbf{k}'$ is along the symmetry axis of the coin and rolling occurs in the direction opposite to that of unit vector $\mathbf{i}' = \mathbf{j}' \times \mathbf{k}'$. Finally, unit vector $\mathbf{k}$ points in the vertical direction and g is gravity.



a. The condition of perfect rolling means that the velocity in contact point P is zero. Use this to show that

$$\mathbf{v}_{cm} = -\mathbf{i}' a\omega_{z'} + \mathbf{k}' a\omega_{x'}$$

where $\omega_{x'} = \boldsymbol{\omega} \cdot \mathbf{i}'$ and $\omega_{z'} = \boldsymbol{\omega} \cdot \mathbf{k}'$.

b. The angular velocity components are given by

$$\omega_{x'} = \dot{\theta}\,, \qquad \omega_{y'} = \dot{\phi}\sin\theta\,, \qquad \omega_{z'} = \dot{\psi} + \dot{\phi}\cos\theta\,,$$

with θ, ϕ and ψ the Eulerian angles. The meaning of θ is given in the figure. Give the definition of angles ϕ and ψ and make a figure in which you sketch ϕ and ψ.

c. Use the Lagrange formalism to show that the equations for the rolling coin read

$$(I + ma^2)\ddot{\theta} = I\dot{\phi}^2 \sin\theta\cos\theta - (I_s + ma^2)S\dot{\phi}\sin\theta - mga\cos\theta\,,$$
$$\frac{d}{dt}\left[I\dot{\phi}\sin^2\theta + (I_s + ma^2)S\cos\theta\right] = 0\,,$$
$$\frac{dS}{dt} = 0\,,$$

with $S = \dot{\psi} + \dot{\phi}\sin\theta$.

d. Note that $\theta = \pi/2$ (upright rolling coin), $\phi = 0$ and S= constant is a solution of the equations of motion in item c.
Under what condition(s) is this a stable solution?

<u>Hint</u>: substitute $\theta = (\pi/2) + \theta', \phi = \phi'$, with $\theta' \ll 1$ and $\phi' \ll 1$, in the equations of motion and maintain only terms that are linear in θ and in ϕ'.

See next page for problem 3

Solution

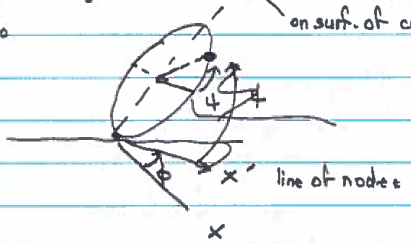
a. In P: $\vec{v} = \vec{v}_{cm} + \vec{\omega} \times \vec{OP} = 0$

From the fig.: $\vec{OP} = -\vec{j}'a \quad \rightarrow \quad \vec{v}_{cm} = \begin{vmatrix} \vec{i}' & \vec{j}' & \vec{k}' \\ \omega_{x'} & \omega_{y'} & \omega_{z'} \\ 0 & a & 0 \end{vmatrix} = \ldots$

b. ϕ is angle between x & x′-axis (line of nodes, intersection of surface of coin and x-y plane) and x-axis

ψ is angle between lon perp. and fixed axis on coin on surf. of coin

So



c. $T = \frac{1}{2} I(\omega_{x'}^2 + \omega_{y'}^2) + \frac{1}{2} I_s \omega_{z'}^2 + \frac{1}{2} m \vec{v}_{cm} \cdot \vec{v}_{cm}$

Using a, b

$$T = \frac{1}{2}(I + ma^2)\dot{\theta}^2 + \frac{1}{2} I (\dot{\phi} \sin\theta)^2 + \frac{1}{2}(I_s + ma^2)(\dot{\phi}\cos\theta + \dot{\psi})^2$$

$$V = mg z_{cm} = mga \sin\theta$$

Now

$$L = T - V$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad \text{where } q_1 = \theta,\ q_2 = \phi,\ q_3 = \psi$$

d. $(I + ma^2)\ddot{\theta}' = -(I_s + ma^2) S \dot{\phi}' + mga\,\theta'$

$I \dot{\phi}' - (I_s + ma^2) S \dot{\theta} = 0$

Subst. second eq. into first equ.

$\rightarrow \quad I(I + ma^2)\ddot{\theta}' + \left[(I_s + ma^2)^2 S^2 - Imga\right]\theta' = 0$

If term between straight brackets > 0 ~ harmonic oscillator

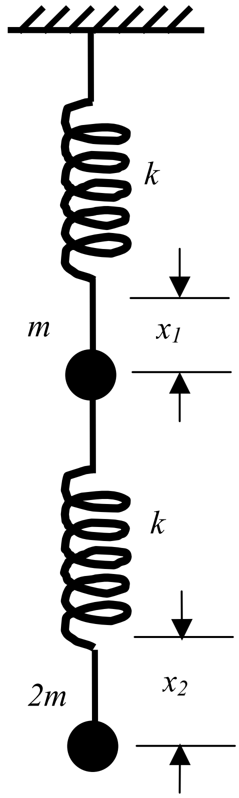
otherwise solutions grow linearly (when $= 0$) or exp. (when < 0)

So $(I_s + ma^2) S^2 > Imga$

Alt. subst. sol. $\theta' \sim \mathrm{Re}\{\theta_0 e^{\lambda t}\}$, $\phi' \sim \mathrm{Re}\{\phi_0 e^{\lambda t}\}$ into equ. for θ', ϕ'

Problem 3

A light elastic spring of stiffness K is clamped at its upper end and supports a particle of mass m at its lower end. A second spring of stiffness K is fastened to the particle and, in turn, supports a particle of mass $2m$ at its lower end. Note: the system in its equilibrium configuration is subject only to gravitational force.



a. Find the normal frequencies of the system for vertical oscillations about the equilibrium configuration.

b. Find the normal coordinates.
If you have no answer of item a, describe the method to find these coordinates.

c. Determine the general solution for $x_1(t), x_2(t)$.
If you have no answer to item b, describe the method to find this solution.

END

Solution

a. The kinetic and potential energies are

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 \tag{1}$$

$$V = \frac{1}{2}Kx_1^2 + \frac{1}{2}K(x_2 - x_1)^2 \tag{2}$$

and therefore we can build the Lagrangian $L = T - V$,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m\ddot{x}_1 \qquad \frac{\partial L}{\partial x_1} = -Kx_1 + K(x_2 - x_1) \tag{3}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = 2m\ddot{x}_2 \qquad \frac{\partial L}{\partial x_2} = -K(x_2 - x_1) \tag{4}$$

Thus we get

$$\begin{vmatrix} -m\omega^2 + 2K & -K \\ -K & -2m\omega^2 + K \end{vmatrix} = 0 \qquad \rightarrow \qquad 2m^2\omega^4 - 5mK\omega^2 + 2K^2 + K^2 = 0 \tag{5}$$

That gives the normal frequencies:

$$\omega^2 = \frac{5 \pm \sqrt{17}}{4}(\frac{K}{m}) \tag{6}$$

b. The equations for the eigenvectors are then:

$$\begin{pmatrix} -m\omega^2 + 2K & -K \\ -K & -2m\omega^2 + K \end{pmatrix} \begin{pmatrix} a_{1,j} \\ a_{2,j} \end{pmatrix} = 0 \tag{7}$$

Inserting $\omega_1^2 = \frac{5+\sqrt{17}}{4}(\frac{K}{m})$ in it for the anti-symmetric mode we get:

$$[\frac{-5 + \sqrt{17}}{4}K + 2K]a_{11} = Ka_{21} \qquad a_{21} = \frac{3 - \sqrt{17}}{4}a_{11} \tag{8}$$

Letting $a_{11} = 1$, then $a_{21} = -0.281$. Similairly, inserting $\omega_2^2 = \frac{5-\sqrt{17}}{4}(\frac{K}{m})$ for the symmetric mode we get:

$$[\frac{-5 - \sqrt{17}}{4}K + 2K]a_{12} = Ka_{22} \qquad a_{22} = \frac{3 + \sqrt{17}}{4}a_{12} \tag{9}$$

Letting $a_{12} = 1$, then $a_{22} = 1.781$.

c. Finally the two normal modes are:

$$Q_1 = \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} cos(\omega_1 t - \delta_1) = 0 \tag{10}$$

$$Q_2 = \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix} cos(\omega_2 t - \delta_2) = 0 \tag{11}$$

from which we can write:

$$x_1(t) = a_{1,1}cos(\omega_1 t - \delta_1) - a_{1,2}cos(\omega_2 t - \delta_2) \tag{12}$$

$$x_2(t) = a_{1,2}cos(\omega_1 t - \delta_1) + a_{22}cos(\omega_2 t - \delta_2) \tag{13}$$

Equation sheet Advanced Mechanics for final exam (version 2019/2020)

A1. Goniometric relations:

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

A2. Spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dxdydz = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$
$$\mathbf{v} = \mathbf{e}_r\,\dot{r} + \mathbf{e}_\theta\,r\dot{\theta} + \mathbf{e}_\phi\,r\dot{\phi}\sin\theta$$
$$\mathbf{a} = \mathbf{e}_r(\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2) + \mathbf{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) + \mathbf{e}_\phi(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\,\sin\theta + 2r\dot{\theta}\dot{\phi}\,\cos\theta)$$

A3. Cylindrical coordinates R, ϕ, z:

$$x = R\cos\phi, \qquad y = R\sin\phi, \qquad z = z$$
$$dxdydz = R\,dR\,d\phi\,dz$$
$$\mathbf{v} = \mathbf{e}_R\,\dot{R} + \mathbf{e}_\phi\,R\dot{\phi} + \mathbf{e}_z\,\dot{z}$$
$$\mathbf{a} = \mathbf{e}_R(\ddot{R} - R\dot{\phi}^2) + \mathbf{e}_\phi\,(2\dot{R}\dot{\phi} + R\ddot{\phi}) + \mathbf{e}_z\,\ddot{z}$$

A4. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

A5. $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$

A6. $\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{Q}$

B1. Noninertial reference frames:

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0$$
$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0$$

C1. Systems of particles:

$$\sum_i \mathbf{F}_i = \frac{d\mathbf{p}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \mathbf{N}$$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{r}_{\text{cm}} \times m\mathbf{v}_{cm} + \sum_i \bar{r}_i \times m_i\bar{v}_i$
where $\bar{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}_{cm}, \bar{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_{cm}$

C3. Equations of motion for 2-particle system with central force:

$$\mu\frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R}$$

with $\mu = m_1m_2/(m_1+m_2)$ the reduced mass, $\mathbf{R}$ relative position vector.

C4. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$$

with $\mathbf{V}$ velocity of Δm relative to m.

D1. Moment of inertia tensor:

$$\mathbf{I} = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i)\,\mathbf{1} - \sum_i m_i \mathbf{r}_i\,\mathbf{r}_i$$

D2. Moment of inertia about an arbitrary axis: $I = \tilde{\mathbf{n}}\,\mathbf{I}\,\mathbf{n} = mk^2$

D3. Formulation for sliding friction: $F_P \;=\; \mu_k\, F_N$

D4. Impulse and rotational impulse: $\mathbf{P} = \int \mathbf{F}dt = m\Delta\mathbf{v}_{cm}$, $\quad \int N dt = P\, l$
with l the distance between line of action and the fixed rotation axis.

E1. Transformation rule components of a real cartesian tensor, rank p, dimension N:

$$T'_{i_1 i_2 \ldots i_p} = \alpha_{i_1 j_1}\alpha_{i_2 j_2}\ldots\alpha_{i_p j_p} T_{j_1 j_2 \ldots j_p}$$

F1. Euler equations: $N_1 = I_1\,\dot{\omega}_1 + \omega_2\omega_3\,(I_3 - I_2)$
(other equations follow by cyclic permutation of indices)

G1. Lagrange's equations (first kind):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + \lambda_k \frac{\partial f_k}{\partial q_i}$$

with $f_k(q_1, q_2, \ldots, q_n, t) = 0$ constraints.

G2. Hamilton's variational principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

G3. Hamiltonian function:

$$H = p_i \dot{q}_i - L$$

G4. Hamilton's canonical equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$
