MID-EXAM ADVANCED MECHANICS, 12 DECEMBER 2019, 13:30-15:30 hours

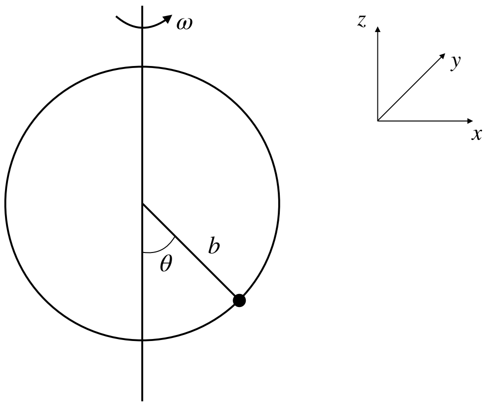
Three problems (all items have a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets.

Problem 1

A point mass m is threaded on a frictionless circular wire hoop of radius b. The hoop lies in a vertical plane, which rotates about the hoop's vertical diameter with a constant angular velocity ω. The position of the point mass is specified by the angle θ measured up from the vertical (see figure).



a. Draw all the forces (physical and inertial) acting on the point in a reference frame rotating with the hoop. Be clear about the names and directions of these forces.

b. Show that the equations of motion (in the same reference frame) are, in polar coordinates,

$$\ddot{r} = 0$$
$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{b}\right)\sin\theta$$

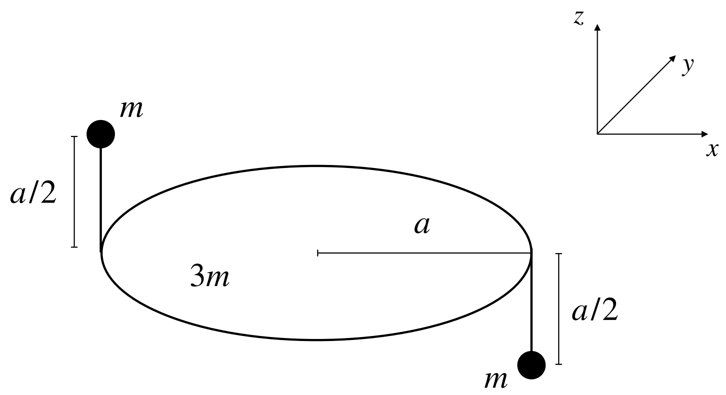
c. From now on we are going to consider only the second equation of motion, as there is no motion in the radial direction. It is easy to prove that $\theta^* = 0$ is an equilibrium position for the point mass. Find the other equilibria (in case they exist), or demonstrate that there are no other equilibria.

d. Consider an initial condition $\theta_0 << 1$ and $\dot{\theta}(t = 0) = 0$. Derive the solution of the equation of motion.

See next page for problem 2

Problem 2

Two point masses m are, by means of rigid massless rods, connected to the edge of a flat disk (radius a and mass $3m$, homogeneous mass distribution ρ). The length of each rod is $a/2$ (see situation sketch). Choose the x- and y-axis in the plane of the disk, the z-axis perpendicular to the disk and take as the origin the center of mass of the disk.



a. Demonstrate that the moment of inertia tensor of this object, in the given coordinate system, can be written as

$$\begin{pmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{pmatrix}$$

and express the components I_{xx}, I_{yy}, I_{zz} and I_{xz} in terms of a and m.

b. Calculate the angular momentum vector of the object in the case that it rotates about the z-axis with angular velocity ω.

c. Calculate the torque that is required to maintain the rotation about the z-axis.

d. Find the principal axes of rotation of the object, as well as the corresponding principal moments of inertia. Explain the physical meaning of principal axes.

See next page for problem 3

Problem 3

Consider a central force $\mathbf{F} = f(r)\frac{\mathbf{r}}{r}$ and velocity vector $\mathbf{v}$ in $\mathbb{R}^3$.

a. Find the components of the antisymmetric part of dyad $\mathbf{F}\mathbf{v}$.

b. Calculate $\varepsilon_3 \,\vdots\, \varepsilon_3$,
i.e., the three-fold contraction of the ε_3 tensor with itself.
Explain how you obtain your answer.

c. Calculate Grad $\mathbf{F}$.

END

Equation sheet Advanced Mechanics for mid-term exam (version 2019)

A1. Goniometric relations:

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

A2. Spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dxdydz = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$
$$\mathbf{v} = \mathbf{e}_r\,\dot{r} + \mathbf{e}_\theta\,r\dot{\theta} + \mathbf{e}_\phi\,r\dot{\phi}\sin\theta$$
$$\mathbf{a} = \mathbf{e}_r(\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2) + \mathbf{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)$$
$$+ \mathbf{e}_\phi(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\,\sin\theta + 2r\dot{\theta}\dot{\phi}\,\cos\theta)$$

A3. Cylindrical coordinates R, ϕ, z:

$$x = R\cos\phi, \qquad y = R\sin\phi, \qquad z = z$$
$$dxdydz = R\,dR\,d\phi\,dz$$
$$\mathbf{v} = \mathbf{e}_R\,\dot{R} + \mathbf{e}_\phi\,R\dot{\phi} + \mathbf{e}_z\,\dot{z}$$
$$\mathbf{a} = \mathbf{e}_R(\ddot{R} - R\dot{\phi}^2) + \mathbf{e}_\phi\,(2\dot{R}\dot{\phi} + R\ddot{\phi}) + \mathbf{e}_z\,\ddot{z}$$

A4. $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

A5. $$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$$

A6. $$\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{Q}$$

B1. Noninertial reference frames:

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0$$
$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0$$

C1. Systems of particles:

$$\sum_i \mathbf{F}_i = \frac{d\mathbf{p}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \mathbf{N}$$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{r}_{\text{cm}} \times m\mathbf{v}_{cm} + \sum_i \bar{r}_i \times m_i\bar{v}_i$
where $\bar{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}_{cm}, \bar{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_{cm}$

C3. Equations of motion for 2-particle system with central force:

$$\mu\frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R}$$

with $\mu = m_1 m_2/(m_1 + m_2)$ the reduced mass, $\mathbf{R}$ relative position vector.

C4. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$$

with $\mathbf{V}$ velocity of Δm relative to m.

D1. Moment of inertia tensor:

$$\mathbf{I} = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i)\,\mathbf{1} - \sum_i m_i \mathbf{r}_i\,\mathbf{r}_i$$

D2. Moment of inertia about an arbitrary axis: $I = \tilde{\mathbf{n}}\,\mathbf{I}\,\mathbf{n} = mk^2$

D3. Formulation for sliding friction: $F_P = \mu_k\,F_N$

D4. Impulse and rotational impulse: $\mathbf{P} = \int \mathbf{F}dt = m\Delta\mathbf{v}_{cm}$, $\int N dt = P\,l$
with l the distance between line of action and the fixed rotation axis.

E1. Transformation rule components of a real cartesian tensor, rank p, dimension N:

$$T'_{i_1 i_2 \ldots i_p} = \alpha_{i_1 j_1}\alpha_{i_2 j_2}\ldots\alpha_{i_p j_p}T_{j_1 j_2 \ldots j_p}$$

F1. Euler equations: $N_1 = I_1\,\dot{\omega}_1 + \omega_2\omega_3\,(I_3 - I_2)$
(other equations follow by cyclic permutation of indices)
