SAMPLE MID-EXAM ADVANCED MECHANICS, DECEMBER 2019, time: 2 hours

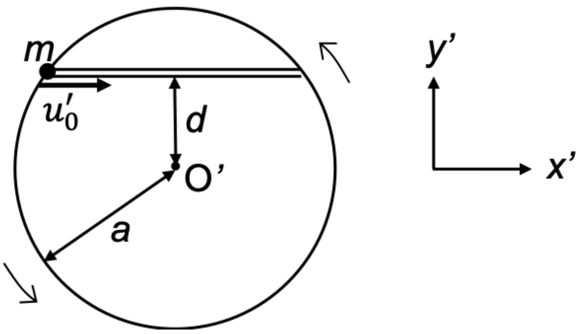
Three problems (all items have a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets.

Problem 1

Consider a circular disk (radius a) that is rotating clockwise about a vertical axis through the middle O' of the disk with constant angular velocity ω. On this disk, a thin, hollow, straight and rigid tube is mounted that has a minimum distance d to the rotation axis (see figure).
At time $t = 0$ a mass point m is released at one end of the tube with an initial velocity u_0' (with respect to the rotating frame) into the tube.



a. Show that the equations of motion for the mass point in the $x' - y'$-plane rotating with the disk read

$$\ddot{x}' = \alpha^2 x' , \qquad\qquad 0 = F_N + \beta\dot{x}' + \gamma ,$$

where F_N is a reactive force and α, β, γ are constants.
Express α, β, γ in terms of the given parameters.

b. Explain physically why there is in general a nonzero reactive force F_N.
Discuss also the direction of this force.

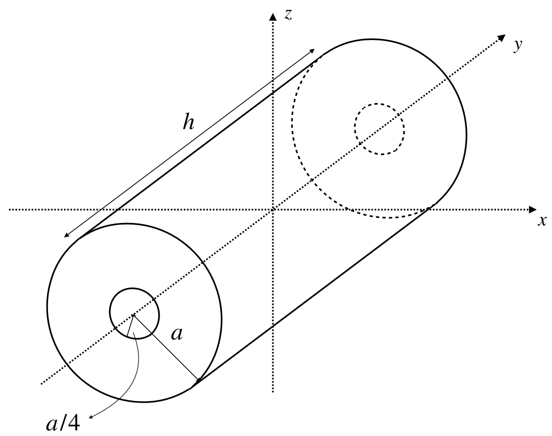
c. Find the path $x'(t)$ of the mass point.

d. Derive and use the energy balance to show that the the mass point can only reach the other end of the tube if its total energy is positive.
Explain the physical meaning of this result.

See next page for problem 2

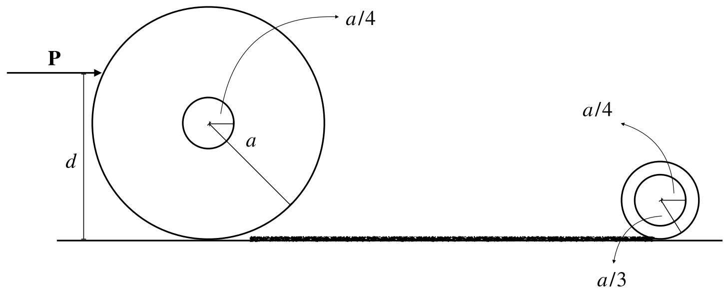
Problem 2

A carpet of mass M is rolled into a hollow cylinder of internal radius $a/4$, external radius a and height h.



a. Choose the principal axes of the cylinder as coordinate axes (see figure above). Determine the components of the moment of inertia tensor with respect to this coordinate system. Show that the chosen axes correspond to the principal axes.

b. Once the carpet is put on the floor, in order to unroll it, a kick is given to it in a direction perpendicular to the axis of the cylinder. The kick can be represented as an horizontal impulsive force $\mathbf{P}$. What is the height d at which the force needs to be applied, for the carpet to start rolling without sliding?



c. The carpet material has a certain thickness (to be assumed negligible), and, as it starts to unroll, the rolled part of the carpet will still be in the shape of a hollow cylinder, although its (external) radius will reduce in time. The unrolled part stays on the ground. The total energy of the system is conserved during its motion. Why is that the case? What about the angular momentum calculated respect to the center of the cylinder?

d. Determine the velocity of the centre of mass and the angular velocity of the carpet when the radius of the unrolled part becomes $a/3$. You don't need to finalise all the calculations, but rather mention all the equations needed to obtain the result.

See next page for problem 3

Problem 3

Consider the moment of inertia tensor $\mathbf{I}$ for a single mass point in $\mathbb{R}^3$.

a. Write it as the sum of an isotropic tensor $\hat{\mathbf{I}}$ and a tensor $\mathbf{I}'$ with zero trace.
   Explain how you obtain your answer.

b. Calculate the vector divergence of $\mathbf{I}$.

c. Determine the rank and components of tensor $\mathbf{T} = \varepsilon_3 : \mathbf{I}$.
   Explain how you obtain your answer.

END

Equation sheet Advanced Mechanics
for mid-term exam (version 2019)

A1. Goniometric relations:

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

A2. Spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dxdydz = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$
$$\mathbf{v} = \mathbf{e}_r\,\dot{r} + \mathbf{e}_\theta\,r\dot{\theta} + \mathbf{e}_\phi\,r\dot{\phi}\sin\theta$$
$$\mathbf{a} = \mathbf{e}_r(\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2) + \mathbf{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)$$
$$+ \mathbf{e}_\phi(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\,\sin\theta + 2r\dot{\theta}\dot{\phi}\,\cos\theta)$$

A3. Cylindrical coordinates R, ϕ, z:

$$x = R\cos\phi, \qquad y = R\sin\phi, \qquad z = z$$
$$dxdydz = R\,dR\,d\phi\,dz$$
$$\mathbf{v} = \mathbf{e}_R\,\dot{R} + \mathbf{e}_\phi\,R\dot{\phi} + \mathbf{e}_z\,\dot{z}$$
$$\mathbf{a} = \mathbf{e}_R(\ddot{R} - R\dot{\phi}^2)\ + \mathbf{e}_\phi\,(2\dot{R}\dot{\phi} + R\ddot{\phi}) + \mathbf{e}_z\,\ddot{z}$$

A4. $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

A5. $$(\mathbf{A} \times \mathbf{B})\cdot\mathbf{C} = (\mathbf{B} \times \mathbf{C})\cdot\mathbf{A} = (\mathbf{C} \times \mathbf{A})\cdot\mathbf{B}$$

A6. $$\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{Q}$$

B1. Noninertial reference frames:

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0$$
$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0$$

C1. Systems of particles:

$$\sum_i \mathbf{F}_i = \frac{d\mathbf{p}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \mathbf{N}$$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{r}_{\text{cm}} \times m\mathbf{v}_{cm} + \sum_i \bar{r}_i \times m_i\bar{v}_i$
where $\bar{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}_{cm}, \bar{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_{cm}$

C3. Equations of motion for 2-particle system with central force:

$$\mu\frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R}$$

with $\mu = m_1 m_2/(m_1 + m_2)$ the reduced mass, $\mathbf{R}$ relative position vector.

C4. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$$

with $\mathbf{V}$ velocity of Δm relative to m.

D1. Moment of inertia tensor:

$$\mathbf{I} = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i)\,\mathbf{1} - \sum_i m_i \mathbf{r}_i\,\mathbf{r}_i$$

D2. Moment of inertia about an arbitrary axis: $I = \tilde{\mathbf{n}}\,\mathbf{I}\,\mathbf{n} = mk^2$

D3. Formulation for sliding friction: $F_P = \mu_k F_N$

D4. Impulse and rotational impulse: $\mathbf{P} = \int \mathbf{F} dt = m\Delta\mathbf{v}_{cm}$, $\int N dt = P\,l$
with l the distance between line of action and the fixed rotation axis.

E1. Transformation rule components of a real cartesian tensor, rank p, dimension N:

$$T'_{i_1 i_2 \ldots i_p} = \alpha_{i_1 j_1}\alpha_{i_2 j_2}\ldots\alpha_{i_p j_p} T_{j_1 j_2 \ldots j_p}$$

F1. Euler equations: $N_1 = I_1\,\dot{\omega}_1 + \omega_2\omega_3\,(I_3 - I_2)$
(other equations follow by cyclic permutation of indices)
