SAMPLE MID-EXAM ADVANCED MECHANICS, DECEMBER 2019, time: 2 hours

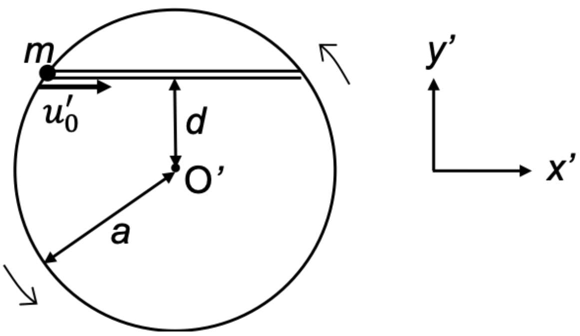
Three problems (all items have a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets.

Problem 1

Consider a circular disk (radius a) that is rotating clockwise about a vertical axis through the middle O' of the disk with constant angular velocity ω. On this disk, a thin, hollow, straight and rigid tube is mounted that has a minimum distance d to the rotation axis (see figure).
At time $t = 0$ a mass point m is released at one end of the tube with an initial velocity u'_0 (with respect to the rotating frame) into the tube.



a. Show that the equations of motion for the mass point in the $x' - y'$-plane rotating with the disk read

$$\ddot{x}' = \alpha^2 x' , \qquad\qquad 0 = F_N + \beta\dot{x}' + \gamma ,$$

where F_N is a reactive force and α, β, γ are constants.
Express α, β, γ in terms of the given parameters.

b. Explain physically why there is in general a nonzero reactive force F_N.
Discuss also the direction of this force.

c. Find the path $x'(t)$ of the mass point.

d. Derive and use the energy balance to show that the the mass point can only reach the other end of the tube if its total energy is positive.
Explain the physical meaning of this result.

Solution

a. Start from $\mathbf{F} = m\mathbf{a}$ in the inertial frame and use the relation between $\mathbf{a}$ and $\mathbf{a}'$ (see equation sheet):

$$m\mathbf{a}' = \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - m\mathbf{A}_0 \,.$$

In the primed rotating coordinate system $\dot{\boldsymbol{\omega}} = 0$ and $\boldsymbol{\omega} = (0, 0, \omega)$ (fixed rotation about vertical axis), $\mathbf{r}' = (x', d, 0)$, $\mathbf{v}' = (u', 0, 0)$, $\mathbf{a}' = (\ddot{x}, 0, 0)$ (mass point in the tube) and $\mathbf{A}_0 = 0$ (fixed origin O'). Furthermore, $\mathbf{F} = (0, F_N, -mg + R)$ (only real force is gravity, F_N and R are reactive forces in the y' and z' direction.
Development of the third and fourth term on the right-hand side (use the determinant rule to evaluate the cross products) yields

$$-2m\boldsymbol{\omega} \times \mathbf{v}' = (0, -2m\omega u', 0) \quad \text{and} \quad -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = (m\omega^2 x', m\omega^2 y', 0).$$

Collection of terms along the x' and y'-axis yields

$$m\ddot{x} = m\omega^2 x' \qquad 0 = F_N - 2m\omega u' + m\omega^2 d \,,$$

so $\quad \alpha = \omega, \quad \beta = -2m\omega, \quad \gamma = m\omega^2 d.$

b. From given equation: $0 = F_N + \beta\dot{x} + \gamma$: reactive force F_N results from joint action of Coriolis force (second term in the equation) and the centrifugal force (third term).
Since the rotation is anticlockwise, the Coriolis force acts to the right on the moving mass point (resulting in a reactive force in the positive y' direction), whilst the centrifugal force is directed away from the rotation axis and results in a negative reactive force. Whether F_N is positive or negative depends on the velocity u', the angular velocity ω and the distance of the mass point to the rotation axis.

c. General solution:

$$x'(t) = A\exp(\alpha^{1/2}t) + B\exp(-\alpha^{1/2}t) \,.$$

Determine A and B from initial conditions $x'(t=0) = -(a^2 - d^2)$ and $\dot{x}'(t=0) = u_0'$. This yields

$$A = -\frac{1}{2}\left[(a^2 - d^2)^{1/2} - \frac{u_0'}{\alpha}\right] \,, \quad B = -\frac{1}{2}\left[(a^2 - d^2)^{1/2} + \frac{u_0'}{\alpha}\right] \,,$$

so

$$x'(t) = -(a^2 - d^2)^{1/2}\cosh(\alpha^{1/2}t) + \frac{u_0'}{\alpha}\sinh(\alpha t) \,.$$

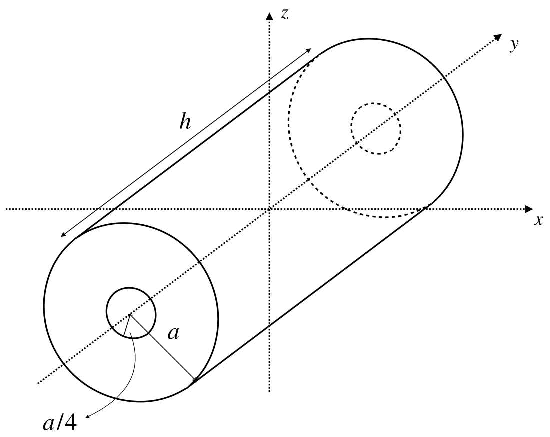
d. Energy equation: multiply momentum equation in x' direction by $m\dot{x}'$. This yields

$$\left(\frac{dE}{dt}\right)_{rot} = 0 \,, \qquad E = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\alpha^2 x'^2,$$

with E the total energy that consists of kinetic energy and potential energy related to the centrifugal force.
For the mass point to reach the other end of the tube, it should have kinetic energy for all positions within the tube. Kinetic energy reaches its minimum T_{min} where potential energy reaches its maximum, i.e., in $x' = 0$: here $T_{min} = E$. The condition is thus $E > 0$. Reason: the mass point has to overcome the centrifugal bulge.

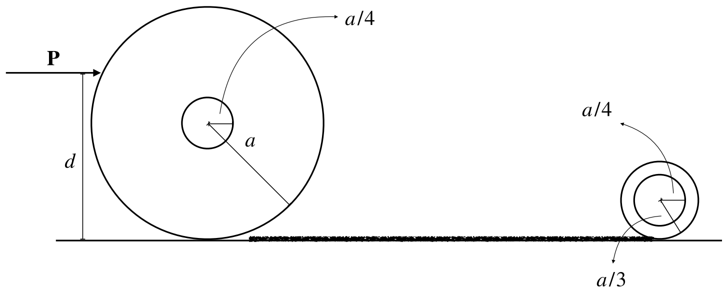
Problem 2

A carpet of mass M is rolled into a hollow cylinder of internal radius $a/4$, external radius a and height h.



a. Choose the principal axes of the cylinder as coordinate axes (see figure above). Determine the components of the moment of inertia tensor with respect to this coordinate system. Show that the chosen axes correspond to the principal axes.

b. Once the carpet is put on the floor, in order to unroll it, a kick is given to it in a direction perpendicular to the axis of the cylinder. The kick can be represented as an horizontal impulsive force $\mathbf{P}$. What is the height d at which the force needs to be applied, for the carpet to start rolling without sliding?



c. The carpet material has a certain thickness (to be assumed negligible), and, as it starts to unroll, the rolled part of the carpet will still be in the shape of a hollow cylinder, although its (external) radius will reduce in time. The unrolled part stays on the ground. The total energy of the system is conserved during its motion. Why is that the case? What about the angular momentum calculated respect to the center of the cylinder?

d. Determine the velocity of the centre of mass and the angular velocity of the carpet when the radius of the unrolled part becomes $a/3$. You don't need to finalise all the calculations, but rather mention all the equations needed to obtain the result.

Solution

a. In order to compute the components of the tensor of inertia of a hollow cylinder, it is convenient to express the mass of the cylinder as a function of and h and the two radii

$$M = \rho h \pi (r_2^2 - r_1^2),$$

where r_2 is the external radius and r_1 the internal one. The mass element, in polar coordinates, can be written as

$$\mathrm{d}m = \rho r \,\mathrm{d}r \,\mathrm{d}y \,\mathrm{d}\theta.$$

We are now ready to calculate the moments of inertia

$$\begin{aligned} I_{yy} &= \int (x^2 + z^2)\,\mathrm{d}m = \rho \int_{-h/2}^{h/2} \int_{r_1}^{r_2} \int_0^{2\pi} r^2 r \,\mathrm{d}y \,\mathrm{d}r \,\mathrm{d}\theta \\ &= 2\pi\rho h \left(\frac{r_2^4}{4} - \frac{r_1^4}{4} \right) = \frac{1}{2} M (r_2^2 + r_1^2), \end{aligned}$$

$$\begin{aligned} I_{xx} &= \int (y^2 + z^2)\,\mathrm{d}m = \rho \int_{-h/2}^{h/2} \int_{r_1}^{r_2} \int_0^{2\pi} (y^2 + r^2 \sin^2\theta)\, r \,\mathrm{d}y \,\mathrm{d}r \,\mathrm{d}\theta \\ &= 2\pi\rho \left(\frac{r_2^2}{2} - \frac{r_1^2}{2} \right) \left(\frac{(h/2)^3}{3} - \frac{(-h/2)^3}{3} \right) + \rho h \left(\frac{r_2^4}{4} - \frac{r_1^4}{4} \right) \pi \\ &= \frac{1}{12} M h^2 + \frac{1}{4} M (r_2^2 + r_1^2), \end{aligned}$$

$$I_{xx} = I_{zz}.$$

To prove the last equation, we only need to notice that

$$\int_0^{2\pi} \sin^2\theta \,\mathrm{d}\theta = \int_0^{2\pi} \cos^2\theta \,\mathrm{d}\theta.$$

The products of inertia I_{xy} and I_{yz} are zero, as they both contain an integral over the interval $[0, 2\pi]$ of either $\cos\theta$ or $\sin\theta$. The term I_{xz} contains an integral of the function $\sin\theta\cos\theta = \sin(2\theta)$, which is also zero. In conclusion, the moment of inertia tensor for a hollow cylinder reads

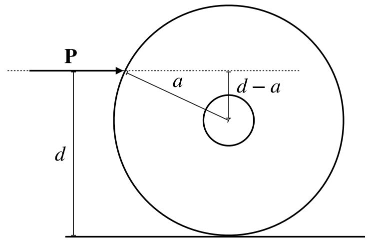
$$\mathbf{I} = \frac{1}{2} M \begin{pmatrix} \frac{1}{6}h^2 + \frac{1}{2}(r_2^2 + r_1^2) & 0 & 0 \\ 0 & r_2^2 + r_1^2 & 0 \\ 0 & 0 & \frac{1}{6}h^2 + \frac{1}{2}(r_2^2 + r_1^2) \end{pmatrix}.$$

The moments of inertia, respectively of the carpet completely unrolled (I_1) and of the carpet in the final position (I_2) (see item c), relative to a rotation about the $y-$axis are

$$I_1 = \frac{17}{32} M_1 a^2 \qquad I_2 = \frac{25}{288} M_2 a^2,$$

where

$$M_1 = \frac{5}{16} \rho h \pi a^2, \qquad M_2 = \frac{7}{144} \rho h \pi a^2.$$



b. When the kick is given to the carpet, the velocity of its center of mass will change, according to the equation

$$v_{cm} = \frac{P}{M_1},$$

where all the quantities are scalar, as their only component is along the $x-$axis. Similarly, the angular velocity becomes

$$\omega = \frac{P(d-a)}{I_1}.$$

Here, it is used that $(d-a)$ is the distance between the line of action and the rotation axis (see figure above). Since the carpet starts rolling without sliding, there is a relation between angular velocity and velocity of the centre of mass

$$v_{cm} = a\omega.$$

Using the equations above, we can find a condition for d

$$\frac{P(d-a)a}{I_1} = \frac{P}{M_1},$$

which leads to $d = \frac{49}{32}a$.

c. The total energy of the system is composed of kinetic and potential energy. There is no friction in the system, except for a force F_P that provides for the rolling; the work done by this force results in a time change of rotational kinetic energy (compare with the problem in FC section 8.6). Thus, total energy is conserved.

For the conservation of the angular momentum to hold, we need the total torque to be zero during the motion of the carpet. However, the frictional force F_P exerts a torque, because the point of application of this force is on the ground, at a certain distance from the unrolled carpet. Moreover, the gravitational force acting on the rolled part of the carpet exerts a torque. Therefore, the angular momentum is not conserved.

d. In order to find the final velocities, we can make use of the conservation of energy

$$T_{1,\text{rot}} + T_{1,\text{trans}} + V_1 = T_{2,\text{rot}} + T_{2,\text{trans}} + V_2 + V_{\text{unrolled}},$$

where the terms at the left hand side refer to the carpet when completely rolled, the first three terms at the right hand side refer to the part of the carpet which is still rolled in the final position and the last term is the potential energy of the unrolled part (lying on the ground). We know that

$$\begin{aligned} T_{i,\text{rot}} &= \frac{1}{2} I_i \omega_i^2, \qquad i = 1, 2, \\ T_{i,\text{trans}} &= \frac{1}{2} M_i v_{i,cm}^2, \qquad i = 1, 2, \\ V_1 &= M_1\, g a, \\ V_2 &= M_2\, g \frac{a}{3}, \\ V_{\text{unrolled}} &= 0. \end{aligned}$$

Moreover, the conditions of pure rolling read

$$v_{1,cm} = \omega_1 a \qquad v_{2,cm} = \omega_2 \frac{a}{3}.$$

Knowing that the initial velocity is $v_{1,cm} = P/M_1$, we can calculate $v_{2,cm}$ using the equations above and compute $\omega_2 = \frac{v_{2,cm}}{a/3}$.

Problem 3

Consider the moment of inertia tensor $\mathbf{I}$ for a single mass point in $\mathbb{R}^3$.

a. Write it as the sum of an isotropic tensor $\hat{\mathbf{I}}$ and a tensor $\mathbf{I}'$ with zero trace.
   Explain how you obtain your answer.
   Explain how you obtain your answer.

b. Calculate the vector divergence of $\mathbf{I}$.

c. Determine the rank and components of tensor $\mathbf{T} = \varepsilon_3 : \mathbf{I}$.
   Explain how you obtain your answer.

Solution

a. For a single mass point $\mathbf{I} = m(r^2\,\mathbf{1} - \mathbf{rr})$.
Write it as $\mathbf{I} = \hat{\mathbf{I}} + \mathbf{I}'$, where all tensors are real and have rank 2.
The isotropic tensor $\hat{\mathbf{I}} = \mu\mathbf{1}$, where μ is a scalar to be determined and $\mathbf{I}$ the unit tensor.
This implies that

$$\mathbf{I}' = \mathbf{I} - \mu\,\mathbf{1}\,.$$

The condition trace($\mathbf{I}'$)=0 yields

$$\text{trace}(\mathbf{I}) - \text{trace}(\mu\,\mathbf{1}) = 0. \qquad \text{or} \qquad m(3r^2 - r^2) - 3\mu = 0\,.$$

From this, it follows $\mu = \frac{2}{3}mr^2$, hence

$$\hat{\mathbf{I}} = \frac{2}{3}mr^2\,, \qquad \mathbf{I}' = m\left(\frac{1}{3}r^2\,\mathbf{1} - \mathbf{rr}\right)\,.$$

b. The vector divergence of tensor $\mathbf{I}$ is defined as

$$\text{Div}\,\mathbf{I} = \boldsymbol{\nabla}\cdot\mathbf{I}\,.$$

This is a tensor of rank (1+2-2), i.e., rank 1, with components

$$\{\text{Div}\,\mathbf{I}\}_i = \frac{\partial}{\partial x_j}\,I_{ji}\,.$$

Development yields

$$\begin{aligned}\frac{\partial}{\partial x_j}\left\{mr^2\delta_{ji} - mx_jx_i\right\} &= 2mr\frac{\partial r}{\partial x_j}\delta_{ji} - m\frac{\partial x_j}{\partial x_j}x_i - mx_j\frac{\partial x_i}{\partial x_j}\\ &= 2mx_j\delta_{ji} - 3mx_i - mx_j\delta_{ij} = -2mx_i\,.\end{aligned}$$

So $\text{Div}\,\mathbf{I} = -2m\mathbf{r}$.

c. Tensor $\boldsymbol{\varepsilon}_3$ has rank 3, whilst tensor $\mathbf{I}$ has rank 2. Tensor $\mathbf{T}$ results from double contraction of $\boldsymbol{\varepsilon}_3$ and $\mathbf{I}$, so the rank of $\mathbf{T}$ is (3+2-4). Hence, $\mathbf{T}$ is a tensor of rank 1 with components

$$T_i = \varepsilon_{ijk}\,I_{jk}\,.$$

Now use that $\boldsymbol{\varepsilon}_3$ is fully antisymmetric, whilst $\mathbf{I}$ is symmetric. Thus,

$$T_i = \frac{1}{2}\varepsilon_{ijk}\,I_{jk} + \frac{1}{2}\varepsilon_{ikj}\,I_{kj} = \frac{1}{2}\varepsilon_{ijk}\,I_{jk} - \frac{1}{2}\varepsilon_{ijk}\,I_{jk} = 0$$

(in the first step: indices j and k are reversed; in the second step the antisymmetry of $\boldsymbol{\varepsilon}_3$ and the symmetry of $\mathbf{I}$ is used).

So $\mathbf{T}$ is the null vector.

Equation sheet Advanced Mechanics for mid-term exam (version 2019)

A1. Goniometric relations:

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha, \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

A2. Spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dxdydz = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$
$$\mathbf{v} = \mathbf{e}_r\,\dot{r} + \mathbf{e}_\theta\,r\dot{\theta} + \mathbf{e}_\phi\,r\dot{\phi}\sin\theta$$
$$\mathbf{a} = \mathbf{e}_r(\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2) + \mathbf{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)$$
$$+ \mathbf{e}_\phi(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\,\sin\theta + 2r\dot{\theta}\dot{\phi}\,\cos\theta)$$

A3. Cylindrical coordinates R, ϕ, z:

$$x = R\cos\phi, \qquad y = R\sin\phi, \qquad z = z$$
$$dxdydz = R\,dR\,d\phi\,dz$$
$$\mathbf{v} = \mathbf{e}_R\,\dot{R} + \mathbf{e}_\phi\,R\dot{\phi} + \mathbf{e}_z\,\dot{z}$$
$$\mathbf{a} = \mathbf{e}_R(\ddot{R} - R\dot{\phi}^2) + \mathbf{e}_\phi\,(2\dot{R}\dot{\phi} + R\ddot{\phi}) + \mathbf{e}_z\,\ddot{z}$$

A4. $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

A5. $$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$$

A6. $$\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{Q}$$

B1. Noninertial reference frames:

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0$$
$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0$$

C1. Systems of particles:

$$\sum_i \mathbf{F}_i = \frac{d\mathbf{p}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \mathbf{N}$$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{r}_{\text{cm}} \times m\mathbf{v}_{cm} + \sum_i \bar{r}_i \times m_i\bar{v}_i$
where $\bar{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}_{cm}, \bar{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_{cm}$

C3. Equations of motion for 2-particle system with central force:

$$\mu\frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R}$$

with $\mu = m_1m_2/(m_1 + m_2)$ the reduced mass, $\mathbf{R}$ relative position vector.

C4. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$$

with $\mathbf{V}$ velocity of Δm relative to m.

D1. Moment of inertia tensor:

$$\mathbf{I} = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i)\,\mathbf{1} - \sum_i m_i \mathbf{r}_i\,\mathbf{r}_i$$

D2. Moment of inertia about an arbitrary axis: $I = \tilde{\mathbf{n}}\,\mathbf{I}\,\mathbf{n} = mk^2$

D3. Formulation for sliding friction: $F_P = \mu_k F_N$

D4. Impulse and rotational impulse: $\mathbf{P} = \int \mathbf{F}dt = m\Delta\mathbf{v}_{cm}$, $\int N dt = P\,l$
with l the distance between line of action and the fixed rotation axis.

E1. Transformation rule components of a real cartesian tensor, rank p, dimension N:

$$T'_{i_1 i_2 \ldots i_p} = \alpha_{i_1 j_1}\alpha_{i_2 j_2} \ldots \alpha_{i_p j_p} T_{j_1 j_2 \ldots j_p}$$

F1. Euler equations: $N_1 = I_1\,\dot{\omega}_1 + \omega_2\omega_3\,(I_3 - I_2)$
(other equations follow by cyclic permutation of indices)
