FINAL EXAM ADVANCED MECHANICS,
30 January 2020, 13:30-15:30, time: 2 hours

Three problems (all items have a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets and add your name on them.

Problem 1

A point mass m is contrained to move on the surface of a sphere with radius a. The sphere is fixed in space, so it is neither translating nor rotating.
The point mass is subject to a single potential force, such that it has a potential energy

$$V = m\,\gamma\,\sin\theta\,\cos(2\phi - \Omega\,t)\,.$$

Here, γ and Ω are constants, r, θ and ϕ are spherical coordinates and t is time.

a. Show that the kinetic energy of this system is of the form

$$T = A(\phi,\theta)\,p_\theta^2 + B(\phi,\theta)\,p_\phi^2\,,$$

with p_θ and p_ϕ the generalised momenta.
Give explicit expressions for the functions $A(\phi,\theta)$ and $B(\phi,\theta)$.

b. Derive Hamiltonian's canonical equations of this system.

c. Give two advantages and two disadvantages of using the Hamilton formalism with respect to applying the Lagrange formalism.

See next page for problem 2

Solution of problem 1

Exam VKM 19/20

Use spherical coordinates, where $r = a \Rightarrow$

1a From equation sheet :

$$T = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} m a^2 \sin^2\theta \, \dot{\phi}^2$$

Generalised momenta

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}(T - V) = \frac{\partial T}{\partial \dot{\theta}} = m a^2 \dot{\theta}$$

$$p_\phi = \cdots = \frac{\partial T}{\partial \dot{\phi}} = m a^2 \sin^2\theta \, \dot{\phi}$$

So

$$T = \frac{1}{2} m a^2 \left(\frac{p_\theta}{m a^2}\right)^2 + \frac{1}{2} m a^2 \sin^2\theta \left(\frac{p_\phi}{m a^2 \sin^2\theta}\right)^2$$

Hence

$$A = \frac{1}{2 m a^2}, \quad B = \frac{1}{2 m a^2 \sin^2\theta}$$

b. Hamiltonian function from equation sheet, or directly

$$H = T + V$$

since coordinate transform does not depend explicitly on time.

Hamilton equations :

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m a^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m a^2 \sin^2\theta}$$

already found in item a

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{m a^2 \sin^3\theta} - m\gamma \cos\theta \cos(2\phi - \Omega t)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 2 m \gamma \sin\theta \sin(2\phi - \Omega t)$$

c. Advantages :

1) H equations first order, standard form
L equations second order, or first order non standard
2) Physical meaning H clear, meaning L not so clear.

Disadvantages :

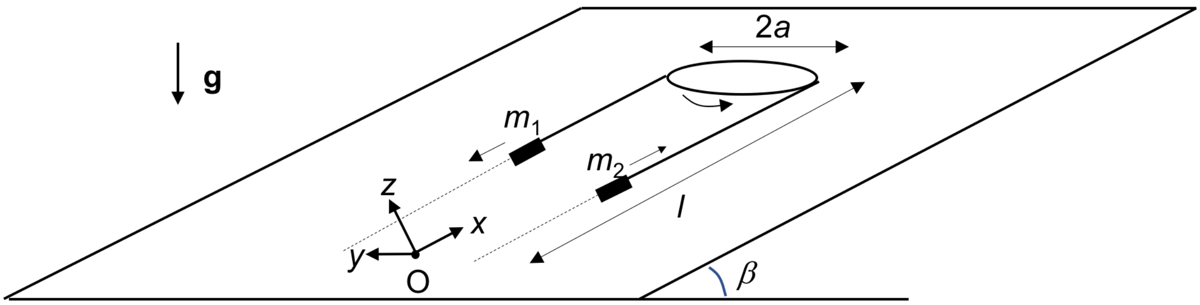
1) H equations only for frictionless systems.
2) H equations can not deal with forces of constraints.

Problem 2

A simple funicular system consists of two cars on a sloping surface, which are connected by a cable that passes over a frictionless pulley (see figure). A motor drives the rotation of the pulley, such that the cars move in opposite directions on straight tracks. In the figure, car 1 is moving downward and car 2 is moving upward.

In this problem, the sloping surface has an angle β with respect to the horizontal, the cars are modelled as point masses m_1 and m_2, the pulley has a moment of inertia I and radius a, such that the tracks are a distance $2a$ apart. The length of each track is l, the position of the centre of the pulley is at $x = l$ and g denotes gravity.

Choose the $x-, y-$ and z-axes and origin O as is shown in the figure (x is along the sloping surface, z is perpendicular to the sloping surface).



a. Three important constraints of this system are

$$f_1 = (x_1 + x_2 - l) = 0\,,$$
$$f_2 = a(\theta - \theta_0) - x_2 = 0\,,$$
$$f_3 = \theta - G(t) = 0\,.$$

Here, x_1 and x_2 are the x-coordinates of car 1 and car 2, respectively, θ is an angle such that $\dot{\theta}$ is the angular velocity of the pulley, θ_0 is a constant and $G(t)$ is a given function of time.
Explain the physical meaning of these three constraints.

b. Show that the kinetic and potential energy of the funicular system are of the form

$$T = \mu_1\,\dot{x}_1^2 + \mu_2\,\dot{x}_2^2 + \mu_3\,\dot{\theta}^2\,,$$
$$V = \nu_1 x_1 + \nu_2 x_2 + \nu_3\theta\,.$$

Express the six constants $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2$ and ν_3 in terms of the given model parameters.

c. Choose x_1, x_2 and θ as generalised coordinates and apply the Lagrange formalism, using the results of items a and b, to derive expressions for
- the tensions in the cable on either side of the pulley;
- the torque that the motor exerts on the pulley to control the motion of the car.

See next page for problem 3

Solution of problem 2

VKM 19/20 exam Problem 2

2a
- $f_1 = 0$: the length of the cable is $L = \ell + \pi a$, L is fixed
  and $L = \ell - x_1 + \pi a + (\ell - x_2)$
  so $x_1 + x_2 = 2\ell + \pi a - L = \ell$
- $f_2 = 0$: velocity $a\dot{\theta}$ at rim of pulley must equal $\dot{x}_2$ (no slip).
- $f_3 = 0$: $\frac{d\theta}{dt}(t)$ is the driver of the angular velocity of the pulley

See equation sheet. There are three constraints, so three Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3$

b.
$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 + \tfrac{1}{2} I \dot{\theta}^2$$
$$V = m_1 g x_1 \sin\beta + m_2 g x_2 \sin\beta$$

So
$$\mu_1 = \tfrac{1}{2} m_1, \quad \mu_2 = \tfrac{1}{2} m_2, \quad \mu_3 = \tfrac{1}{2} I$$
$$v_1 = m_1 g \sin\beta, \quad v_2 = m_2 g \sin\beta, \quad v_3 = 0.$$

Note: the pulley has potential energy, but it is constant.

c.
$$L = T - V$$

Lagrange's equations of the first kind: e.g.
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1} + \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} + \lambda_3 \frac{\partial f_3}{\partial x_1}$$

becomes
$$m_1 \ddot{x}_1 = -m_1 g \sin\beta + \lambda_1$$
$$m_2 \ddot{x}_2 = -m_2 g \sin\beta + \lambda_1 - \lambda_2$$
$$I\ddot{\theta} = \lambda_2 a + \lambda_3$$

So $T_1 = \lambda_1$: tension in cable on left. $T_2 = (\lambda_1 - \lambda_2)$ tension in cable on right,
$N = \lambda_3$: the torque exerted by the motor ($\lambda_2 a$ is torque due to difference in tensions in cable on left and right, which occurs if $m_1 \neq m_2$)

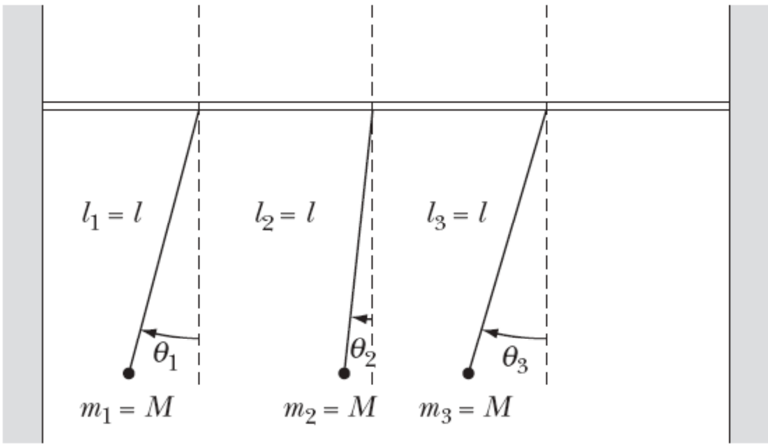
Use constraints to rewrite equations as
$$-m_1 a \ddot{\theta} = -m_1 g \sin\beta + \lambda_1$$
$$m_2 a \ddot{\theta} = -m_2 g \sin\beta + \lambda - \lambda_2$$
$$I\ddot{\theta} = \lambda_2 a + \lambda_3$$

From :
First equation : $T_1 = \lambda_1 = m_1 (g \sin\beta - a\ddot{\theta})$
Second equation : $T_2 = \lambda_1 - \lambda_2 = m_2 (g \sin\beta + a\ddot{\theta})$
Third equation : $N = \lambda_3 = I\ddot{\theta} - \lambda_2 a$

=

Problem 3

Three identical pendulums of mass M and length l are suspended from a slightly elastic, massless rod. The elasticity in the rod brings about a coupling (with constant K) between each pair of masses m_i and m_j, with a corresponding potential energy $V_{ij} = \frac{1}{2}K(x_i - x_j)^2$. Here, x_i and x_j are the horizontal displacements of m_i and m_j with respect to their equilibrium positions.
Consider only the case of small oscillations.



a. One of the eigenfrequencies of the system is $\omega_3 = \left(\frac{g}{l}\right)^{1/2}$.
   Find the other eigenfrequencies of this system.

b. Find the normal modes of oscillation.
   If you have no answer to item a, then describe the method to find the normal modes.

c. Determine the general solution for $\theta_1(t), \theta_2(t), \theta_3(t)$.
   If you have no answer to item b, describe the method to find this solution.

END

Solution problem 3

Three identical pendulums that are coupled through a slightly yielding rod.

a. The kinetic energy of the system is

$$T = \frac{1}{2} M l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 \right) .$$

The potential energy is

$$V = Mg(z_1 + z_2 + z_3) + \frac{1}{2} K((x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2) .$$

In this case, oscillations are small, so

$$x_i = -l \sin \theta_i \simeq l\theta_i , \qquad z\_i = l[1 - \cos \theta_i] \simeq \frac{1}{2} l \theta_i^2 ,$$

hence

$$V = \frac{1}{2} Mgl(\theta_1^2 + \theta_2^2 + \theta_3^2) + \frac{1}{2} K l^2((\theta_1 - \theta_2)^2 + (\theta_1 - \theta_3)^2 + (\theta_2 - \theta_3)^2) .$$

Next, construct the Lagrangian $L = T - V$, derive Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) = \left(\frac{\partial L}{\partial \theta_i} \right)$$

and write them in standard form. This yield the following $\mathbf{M}$ and $\mathbf{K}$ matrices:

$$\mathbf{M} = M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{K} = \begin{pmatrix} \alpha & -K & -K \\ -K & \alpha & -K \\ -K & -K & \alpha \end{pmatrix} ,$$

where $\alpha = (M(g/l) + 2K)$. For obtaining the eigenfrequencies, we have to evaluate the determinant:

$$|K - \omega^2 M| = \begin{vmatrix} \alpha - M\omega^2 & -K & -K \\ -K & \alpha - M\omega^2 & -K \\ -K & -K & \alpha - M\omega^2 \end{vmatrix} = 0 .$$

Introducing $\lambda = (\alpha - M\omega^2)$, this expression can be rewritten as

$$\lambda^3 - 3K^2\lambda - 2K^3 = 0 ,$$

or, using the hint,

$$(\lambda - 2K)[\lambda^2 + 2K\lambda + K^2] = 0 .$$

which yields $\lambda_3 = 2K$ and $\lambda_1 = \lambda_2 = -K$.
Using the definition of λ it follows the three eigenfrequencies

$$\omega_1^2 = \omega_2^2 = \frac{g}{l} + 3\frac{K}{M} , \qquad \omega_3^2 = \frac{g}{l} .$$

b. Having evaluated the eigenfrequencies, we can insert them back into the equations of motion to find the eigenvectors **a** . That is, starting with ω_3:

$$(K_{j,k} - \omega_3^2 M_{j,k})a_{j3} = 0\,.$$

That gives $a_{13} = a_{23} = a_{33} = 1/\sqrt{3}$.

If we repeat the calculation for $\omega_1 = \omega_2$ after a bit of algebra we have

$$\mathbf{a_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{a_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{a_2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

c. The three normal modes are

$$\mathbf{Q}_1 = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \end{pmatrix} \cos(\omega_1 t - \delta_1)\,,$$

$$\mathbf{Q}_2 = \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \end{pmatrix} \cos(\omega_2 t - \delta_2)\,,$$

$$\mathbf{Q}_3 = \begin{pmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \end{pmatrix} \cos(\omega_2 t - \delta_3)\,,$$

where $a_{i,j}$ is the i'th component of eigenvector j. From this, we can construct the general solution:

$$\theta_1(t) = 2A_2 \cos(\omega_2 t - \delta_1) + A_3 \cos(\omega_3 t - \delta_3)\,,$$

$$\theta_2(t) = A_1 \cos(\omega_1 t - \delta_1) - A_2 \cos(\omega_2 t - \delta_2) + A_3 \cos(\omega_3 t - \delta_3)\,,$$

$$\theta_3(t) = -A_1 \cos(\omega_1 t - \delta_1) - A_2 \cos(\omega_2 t - \delta_2) + A_3 \cos(\omega_3 t - \delta_3)\,,$$

with A_1, A_2, A_3 amplitudes and $\delta_1, \delta_2, \delta_3$ phases that depend on the initial conditions.

Equation sheet Advanced Mechanics for final exam (version 2019/2020)

A1. Goniometric relations:

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha,$ $\quad\quad$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha,$ $\quad\quad$ $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

A2. Spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dxdydz = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$
$$\mathbf{v} = \mathbf{e}_r\,\dot{r} + \mathbf{e}_\theta\,r\dot{\theta} + \mathbf{e}_\phi\,r\dot{\phi}\sin\theta$$
$$\mathbf{a} = \mathbf{e}_r(\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2) + \mathbf{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) + \mathbf{e}_\phi(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\,\sin\theta + 2r\dot{\theta}\dot{\phi}\,\cos\theta)$$

A3. Cylindrical coordinates R, ϕ, z:

$$x = R\cos\phi, \qquad y = R\sin\phi, \qquad z = z$$
$$dxdydz = R\,dR\,d\phi\,dz$$
$$\mathbf{v} = \mathbf{e}_R\,\dot{R} + \mathbf{e}_\phi\,R\dot{\phi} + \mathbf{e}_z\,\dot{z}$$
$$\mathbf{a} = \mathbf{e}_R(\ddot{R} - R\dot{\phi}^2) + \mathbf{e}_\phi\,(2\dot{R}\dot{\phi} + R\ddot{\phi}) + \mathbf{e}_z\,\ddot{z}$$

A4. $\quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

A5. $\quad (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$

A6. $\quad \left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{Q}$

B1. Noninertial reference frames:

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0$$
$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0$$

C1. Systems of particles:

$$\sum_i \mathbf{F}_i = \frac{d\mathbf{p}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \mathbf{N}$$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{r}_{\text{cm}} \times m\mathbf{v}_{cm} + \sum_i \bar{r}_i \times m_i\bar{v}_i$
where $\bar{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}_{cm}, \bar{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_{cm}$

C3. Equations of motion for 2-particle system with central force:

$$\mu\frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R}$$

with $\mu = m_1m_2/(m_1 + m_2)$ the reduced mass, $\mathbf{R}$ relative position vector.

C4. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$$

with $\mathbf{V}$ velocity of Δm relative to m.

D1. Moment of inertia tensor:

$$\mathbf{I} = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i)\,\mathbf{1} - \sum_i m_i \mathbf{r}_i\,\mathbf{r}_i$$

D2. Moment of inertia about an arbitrary axis: $I = \tilde{\mathbf{n}}\,\mathbf{I}\,\mathbf{n} = mk^2$

D3. Formulation for sliding friction: $F_P = \mu_k F_N$

D4. Impulse and rotational impulse: $\mathbf{P} = \int \mathbf{F}dt = m\Delta\mathbf{v}_{cm}$, $\int N dt = P\,l$
with l the distance between line of action and the fixed rotation axis.

E1. Transformation rule components of a real cartesian tensor, rank p, dimension N:

$$T'_{i_1 i_2 \ldots i_p} = \alpha_{i_1 j_1}\alpha_{i_2 j_2} \ldots \alpha_{i_p j_p} T_{j_1 j_2 \ldots j_p}$$

F1. Euler equations: $N_1 = I_1\,\dot{\omega}_1 + \omega_2\omega_3\,(I_3 - I_2)$
(other equations follow by cyclic permutation of indices)

G1. Lagrange's equations (first kind):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + \lambda_k \frac{\partial f_k}{\partial q_i}$$

with $f_k(q_1, q_2, \ldots, q_n, t) = 0$ constraints.

G2. Hamilton's variational principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

G3. Hamiltonian function:

$$H = p_i \dot{q}_i - L$$

G4. Hamilton's canonical equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$
