FINAL EXAM ADVANCED MECHANICS,
3 February 2022, 13:30-15:30, time: 2 hours

Three problems (nine items, each with a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets and add your name on them.

Problem 1

The motion of a star, represented as a mass point m, in a cylindrically symmetric galaxy can be described by two cartesian coordinates x and y in the plane of symmetry of the galaxy. Following a famous paper by Hénon & Heiles (1964), the gravitational potential of the galaxy is modelled as

$$\Phi = \frac{1}{2}\alpha(x^2 + y^2) + \beta(x^2 y - \frac{1}{3}y^3),$$

with α and β positive constants.

a. Find the Hamiltonian function of this system.

b. Hamilton's canonical equations have a so-called symplectic structure. Explain what is meant by 'symplectic structure' and what consequence(s) this structure has for the dynamics of the system.

c. Find one equilibrium of this system and determine its stability properties.

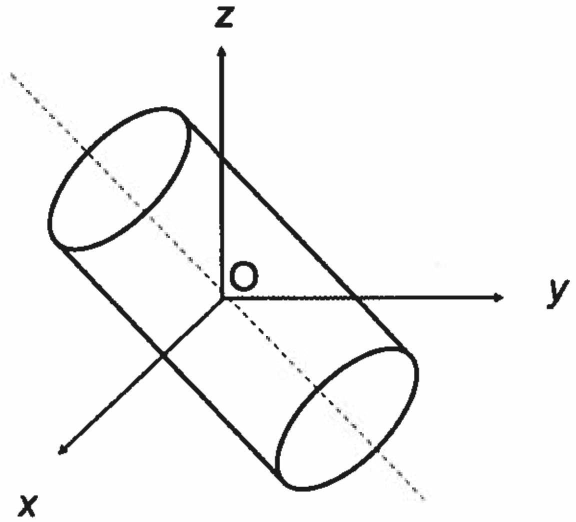
See next page for problem 2

Problem 2

An elongated, cylindrically shaped satellite is undergoing a pure rotation in space (see figure). With respect to its principal axes, the moment of inertia tensor of this object reads

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} ,$$

where $I_1 = I_2 = I$ and $I_3 = I_s$. Here, the third axis is the symmetry axis (the dashed line in the figure). The $x-$, $y-$ and z-axes are defined in the inertial frame and the origin O is the centre of mass of the satellite.



The satellite is subject to a torque

$$\mathbf{N} = -\alpha(\boldsymbol{\omega} \cdot \mathbf{e}_3)\mathbf{e}_3 ,$$

where $\boldsymbol{\omega}$ is the angular velocity vector and $\mathbf{e}_3$ is a unit vector along the symmetry axis of the object (shown as a dashed line in the figure) and α is a positive constant.

a. Describe the physical meaning of I and I_s and argue whether I_s is larger than, equal to, or smaller than I (no calculations needed).

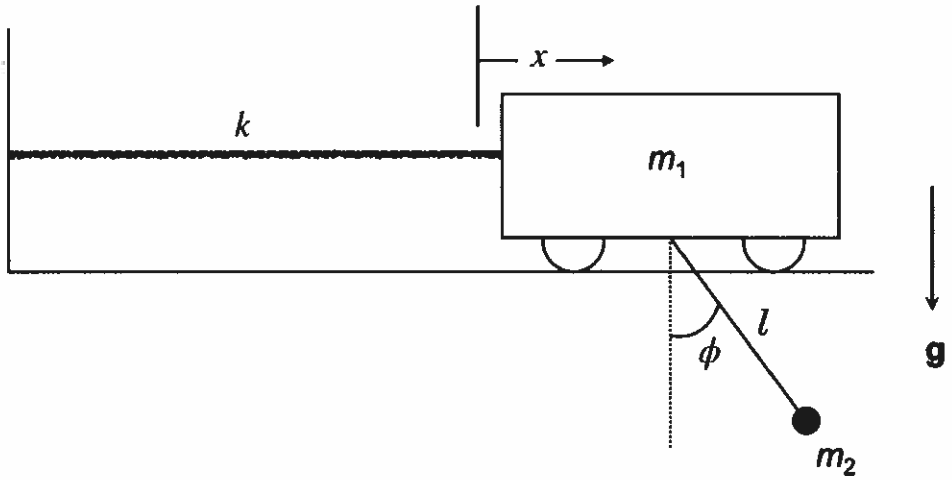
b. Demonstrate, by analysing the angular momentum balance in the co-rotating frame, that for any initial rotation that is not exactly along the symmetry axis, the rotation of the satellite for $t \to \infty$ will be about an axis that is in the plane of symmetry and that axis passes through the centre of mass of the satellite.

c. When viewed in the inertial frame the rotation of the satellite can be described by three angular velocities $\dot{\phi}, \dot{\theta}$ and $\dot{\psi}$, where ϕ, θ and ψ are Euler angles.
Copy the figure to your answer sheet, indicate the Euler angles in that figure and specify the rotation axes that correspond to the angular velocity components $\dot{\phi}, \dot{\theta}$ and $\dot{\psi}$.

See next page for problem 3

Problem 3

A cart (mass m_1) can freely oscillate on horizontal rails on the end of a spring with spring constant k, as shown in the figure below. At the cart, a simple pendulum is mounted (length l, mass m_2), which can oscillate in the plane of the paper. Ignore effects of friction.



a. Show that, for small values of the perturbed position x and angles ϕ, the Lagrangian of this system is of the form

$$L = \sum_{j,k} \frac{1}{2} M_{jk} \dot{q}_j \dot{q}_k - \sum_{j,k} \frac{1}{2} K_{jk} q_j q_k \,.$$

Here, $q_1 = x$ and $q_2 = \phi$.
Find explicit expressions for the coefficient M_{jk} and K_{jk}.

b. Compute the eigenfrequencies of the system that is described by the Lagrangian given in item a for the case that the masses are equal, $m_1 = m_2 = m$ and for $\frac{k}{m} = 2g/l$.
(If you have no answer to item a, use the general expression of the Lagrangian given in item a.)

c. Find the general solution of the system that is described by the Lagrangian of item a.
(If you have no answer to item b, describe in detail the method to obtain this solution.)

END

Equation sheet Advanced Mechanics for mid-term exam (version 2020/21)

A1. Goniometric relations:

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha,$ $\qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha,$ $\qquad \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

A2. Spherical coordinates r, θ, ϕ:

$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$

$dxdydz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$\mathbf{v} = \mathbf{e}_r\, \dot{r} + \mathbf{e}_\theta\, r\dot{\theta} + \mathbf{e}_\phi\, r\dot{\phi}\sin\theta$

$\mathbf{a} = \mathbf{e}_r(\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2) + \mathbf{e}_\theta(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)$
$\qquad + \mathbf{e}_\phi(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)$

A3. Cylindrical coordinates R, ϕ, z:

$x = R\cos\phi, \qquad y = R\sin\phi, \qquad z = z$

$dxdydz = R\, dR\, d\phi\, dz$

$\mathbf{v} = \mathbf{e}_R\, \dot{R} + \mathbf{e}_\phi\, R\dot{\phi} + \mathbf{e}_z\, \dot{z}$

$\mathbf{a} = \mathbf{e}_R(\ddot{R} - R\dot{\phi}^2) + \mathbf{e}_\phi\,(2\dot{R}\dot{\phi} + R\ddot{\phi}) + \mathbf{e}_z\, \ddot{z}$

A4. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

A5. $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$

A6. $\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{Q}$

B1. Noninertial reference frames:

$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0$

$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0$

C1. Systems of particles:

$\sum_i \mathbf{F}_i = \frac{d\mathbf{p}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \mathbf{N}$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{r}_{cm} \times m\mathbf{v}_{cm} + \sum_i \bar{r}_i \times m_i \bar{v}_i$
where $\bar{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}_{cm}, \bar{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_{cm}$

C3. Equations of motion for 2-particle system with central force:

$$\mu\frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R}$$

with $\mu = m_1 m_2/(m_1 + m_2)$ the reduced mass, $\mathbf{R}$ relative position vector.

C4. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m}$$

with $\mathbf{V}$ velocity of Δm relative to m.

D1. Moment of inertia tensor:

$$\mathbf{I} = \sum_i m_i(\mathbf{r}_i \cdot \mathbf{r}_i)\,\mathbf{1} - \sum_i m_i \mathbf{r}_i\,\mathbf{r}_i$$

D2. Moment of inertia about an arbitrary axis: $I = \tilde{\mathbf{n}}\,\mathbf{I}\,\mathbf{n} = mk^2$

D3. Formulation for sliding friction: $F_P = \mu_k F_N$

D4. Impulse and rotational impulse: $\mathbf{P} = \int \mathbf{F}dt = m\Delta\mathbf{v}_{cm}$, $\int N dt = P\,l$
with l the distance between line of action and the fixed rotation axis.

E1. Euler equations: $N_1 = I_1\,\dot{\omega}_1 + \omega_2\omega_3\,(I_3 - I_2)$
(other equations follow by cyclic permutation of indices)

F1. Lagrange's equations (first kind):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + \lambda_k \frac{\partial f_k}{\partial q_i}$$

with $f_k(q_1, q_2, \ldots, q_n, t) = 0$ constraints.

F2. Lagrange's equations (second kind):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

with $f_k(q_1, q_2, \ldots, q_n, t) = 0$ constraints.

F3. Hamilton's variational principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

F4. Hamiltonian function:

$$H = p_i\dot{q}_i - L$$

F5. Hamilton's canonical equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$
