MID-EXAM ADVANCED MECHANICS,
12 December 2024, time: 2 hours

Three problems; all items have a value of 10 points

Remark 1 : Answers may be written in English or Dutch.
Remark 2: Write answers of each problem on separate sheets with your name on them.

Problem 1

A bead of mass m is constrained to move along a massless, frictionless rod of unlimited length. The rod rotates in the horizontal plane with constant angular velocity ω about a vertical axis that passes through the pivot P (see figure below).

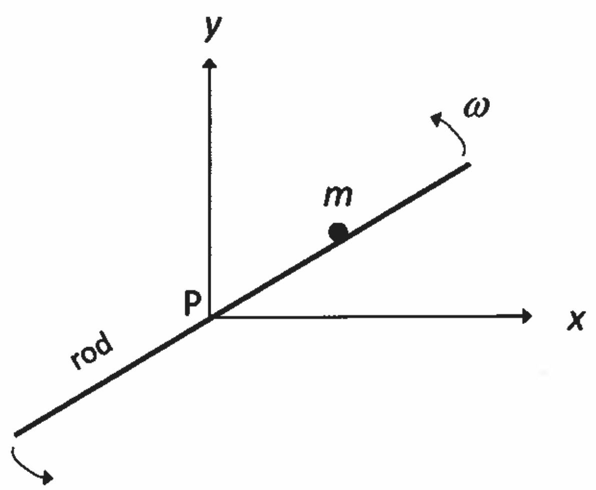


Figure 1: Situation sketch in the horizontal plane.

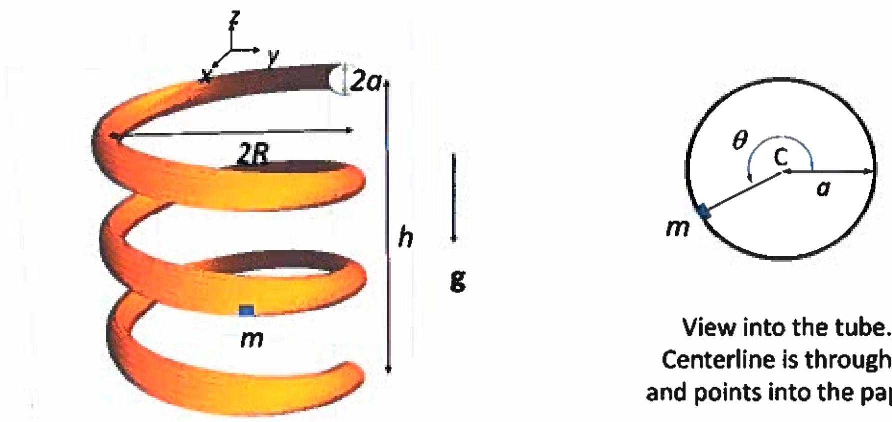
a. Find the Lagrangian of this system, and use the Lagrangian to derive Lagrange's equations.
   Explain the steps that you make.

b. The Lagrange equations can be derived from D'Alembert's principle.
   Write down the mathematical formulation of this principle, including the definition of all variables and parameters that are part of this principle.
   Also, describe the physical meaning of this principle and explain why it is so important.

See next page for Problem 2

Problem 2

A mass point m is sliding down on the frictionless inner surface of a spiralling tube (see figure). The height of the tube is h, the radius of a loop is R and the radius of the circular cross-section of the tube is a.



View into the tube.
Centerline is through C
and points into the paper.

The motion of the mass point can be described by two generalised coordinates: ϕ and θ. Here, ϕ parameterises the centerline of the tube ($\phi = 0$ at $z = 0$ and $\phi = h/b$ at $z = -h$, with b a constant). Generalised coordinate θ is the angle that the line between mass point and the centre of the circular cross-section makes with the x-axis (see right part of the figure).

The z-coordinate of a point on the inner surface of the tube reads

$$z = -b\phi + \frac{R\,a}{\sqrt{R^2 + b^2}} \sin(\theta)\,,$$

Furthermore, the kinetic energy of the mass point reads

$$T = \frac{1}{2}A(\phi,\theta)\,\dot{\phi}^2 + \frac{1}{2}B(\phi,\theta)\,\dot{\theta}^2 + C(\phi,\theta)\,\dot{\phi}\,\dot{\theta}\,,$$

where A, B and C are known functions (you do not need to determine them).

a. Find an explicit expression for the Hamiltonian of this system.
   Its expression should be such that Hamilton's equations can be directly derived from it.

   Hint: Do not spend time on finding the simplest expression of the Hamiltonian.

b. Discuss whether this system has conserved quantities, in particular
   1) generalised momenta;
   2) components of the momentum vector;
   3) components of the angular momentum vector;
   4) energy.

   Explain your answer (no derivations asked).

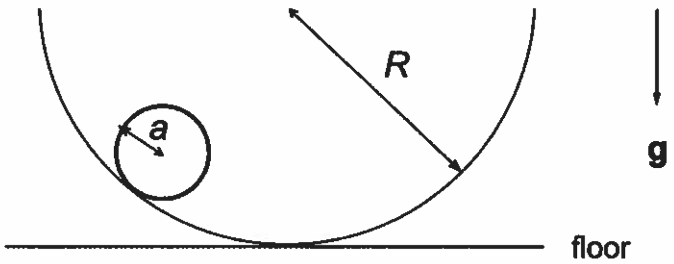
See next page for Problem 3

Problem 3

The figure below shows a hollow cylinder of radius a and uniform surface mass density ϱ. The length of this cylinder in the direction perpendicular to the paper is l.

This cylinder is free to roll along the perfectly rough inner surface of a larger, fixed cylinder of radius $R > a$. The motion of the smaller cylinder is in the plane of the paper and not influenced by air friction.

The system can be characterised by three generalised coordinates: polar coordinates r, ϕ with respect to the centre of the fixed, large cylinder, and the rotation angle θ of the small cylinder with respect to a line in the horizontal plane that passes through its centre of mass. Choose ϕ such that $\phi = 0$ corresponds to the moving cylinder being at its lowest position (its contact point touches the floor).



a. Show that the moment of inertia I of the moving cylinder is given by

$$I = \beta M a^2 ,$$

where M is the mass of the cylinder.
Find coefficient β.

b. The system is subject to two constraints of the type $f_i(r, \phi, \theta) = 0$, where $i = 1, 2$. Specify the functions f_i and explain the physical meaning of the constraints.

c. Derive the modified Lagrange equations of this system.

d. Describe and explain the motion of the small cylinder when it is initially released at an angle $\phi = \phi_0$, with $|\phi_0| \leq \pi$, (without velocity) and describe the physical meaning of the two Lagrange multipliers.

(No need to analyse the modified Lagrange equations, but it may help.)

END

Equation sheet Advanced Mechanics for mid-term exam (version 2024)

A1. Goniometric relations:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \qquad \cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\sin(2\theta) = 2\sin\theta\cos\theta, \qquad \sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

A2. Spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dxdydz = r^2\sin\theta\, dr\, d\theta\, d\phi$$
$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}}$$
$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\hat{\boldsymbol{\theta}}$$
$$+(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,\hat{\boldsymbol{\phi}}$$
$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

A3. Cylindrical coordinates ρ, ϕ, z:

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z$$
$$dxdydz = \rho\, d\rho\, d\phi\, dz$$
$$\mathbf{v} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{z}}$$
$$\mathbf{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\,\hat{\boldsymbol{\phi}} + \ddot{z}\,\hat{\mathbf{z}}$$
$$\nabla f = \hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

A4. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

A5. $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$

B1. Formulation for sliding friction: $F_s = \mu_k F_N$

B2. Impulse and rotational impulse due to a kick:

Impulse $\Delta\mathbf{P} = M\Delta\dot{\mathbf{R}}$, $\qquad \Delta N = \Delta P\, l'$

B3. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{v}_{ex}\dot{m}$$

with $\mathbf{v}_{ex}$ velocity of dm relative to m.

C1. Systems of particles:

$$\sum_{\alpha} \mathbf{F}_{\alpha} = \frac{d\mathbf{P}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \mathbf{\Gamma}$$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_{\alpha} \mathbf{r}'_{\alpha} \times \mathbf{p}'_{\alpha}$

where $\mathbf{r}_{\alpha} = \mathbf{r}_{\alpha} - \mathbf{R}, \quad \mathbf{p}'_{\alpha} = \mathbf{p}_{\alpha} - \mathbf{P}$.

C3. Moment of inertia about an arbitrary axis: $I = \int \varrho\, r_{\perp}^2 dV$

D1. Lagrange's equations (including constraint forces):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} + \lambda_k \frac{\partial f_k}{\partial q_i}$$

D2. Hamilton's variational principle:

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = 0$$

D3. Hamiltonian function:

$$\mathcal{H} = p_i \dot{q}_i - \mathcal{L}$$

D4. Hamilton's canonical equations:

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$
