FINAL EXAM ADVANCED MECHANICS

30 January 2025, time: 2 hours

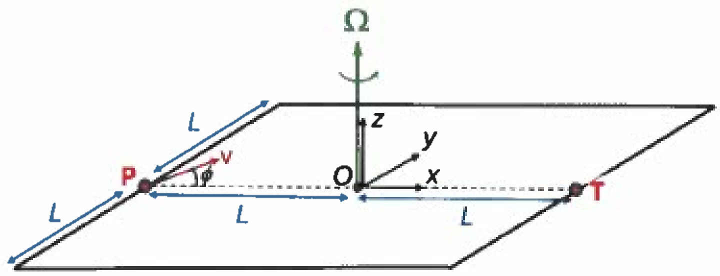
Three problems (all eight items have a value of 10 points)

Remark 1 : Answers may be written in English or Dutch.

Remark 2: Write answers of each problem on separate sheets and add your name on them.

Problem 1

An attraction on a lunapark consists of a rectangular horizontal platform of sizes $2L$ that can rotate anticlockwise about a vertical axis that passes through its middle O (see figure). At the time that a player enters the attraction the platform is at rest. The player is positioned at point P (coordinates $x = -L, y = 0$), so this person is fixed with respect to the platform.



O-xyz is a co-rotating frame

Next, at a specific time the angular velocity of the platform increases monotonically until at time $t = 0$ it has reached its final constant value Ω. At time $t = 0$ the player shoots a disk (modelled as a mass point m) with an initial velocity $v = |\mathbf{v}|$ set by the machine, but the player is free to choose the initial direction ϕ of the disk (ϕ is the angle between velocity vector $\mathbf{v}$ and the positive x-axis). The disk will slide over the horizontal platform without experiencing friction. The aim of the game is to let the disk hits target T, which is on the other side of the platform (coordinates $x = L, y = 0$, see figure).

a. Write down the most general form of Newton's second law in a non-inertial frame, name all the forces that it contains, and use it to show that the motion of the sliding mass point in the xy-plane is described by the equations

$$\ddot{x} - \alpha\dot{y} - \beta x = 0\,, \qquad \ddot{y} + \alpha\dot{x} - \beta y = 0\,.$$

Express coefficients α and β in terms of the given model parameters.

b. Find the solutions of the equations of motion of item b that obey the initial conditions. After that, describe how from these solutions the angle ϕ can be determined that will result in mass point m hitting target T.

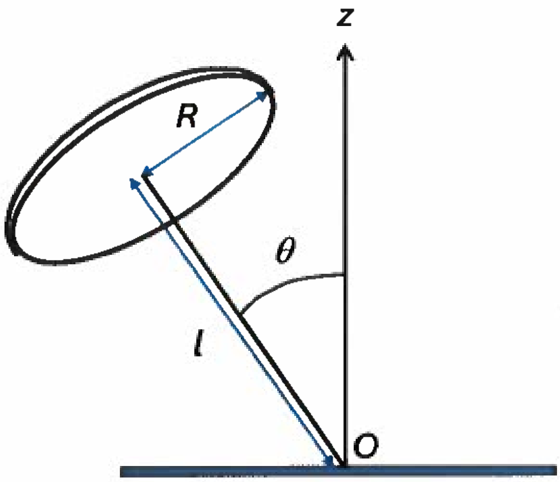
Hint: Use a complex function $x + iy$.

See next page for Problem 2

Problem 2

A top consists of a flat circular disk of radius R and mass M that is mounted on top of a stiff, massless rod that has length l. This rigid body has principal moments of inertia $\lambda_1 = \frac{1}{4}MR^2$ for rotations about any axis in the plane of symmetry, and $\lambda_3 = \frac{1}{2}MR^2$ for rotations about the symmetry axis.

The end of the rod that is not connected to the disk is placed at a fixed point O of a horizontal floor under an angle $\theta(t = 0) = \theta_0$ with respect to the vertical (see figure), and subsequently the body is given the initial conditions $\dot{\theta}(t = 0) = 0, \dot{\phi}(t = 0) = \dot{\phi}_0$ and $\dot{\psi}(t = 0) = \dot{\psi}_0$. Here, θ, ϕ and ψ are the Euler angles. Ignore effects of friction.



a. The equation of motion for angle θ of this top reads

$$\ddot{\theta} = \dot{\phi}^2 \sin\theta\cos\theta - 2\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta + \frac{4gl}{R^2}\sin\theta\,.$$

Find the Lagrangian from which this equation is derived (its expression should only contain parameters that are in the given equation).

Describe the steps that you made and, when you introduce new variables that are not mentioned in this problem, explain them.

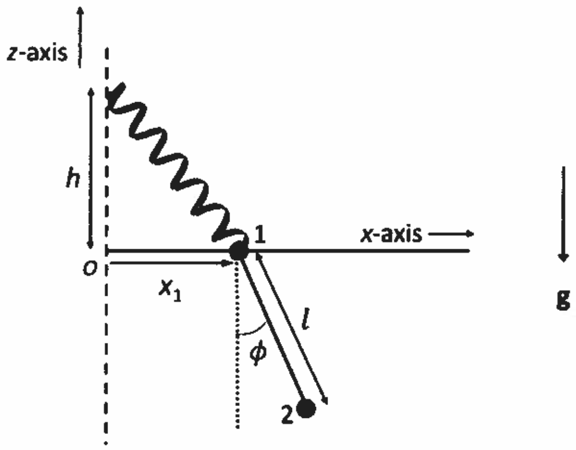
b. Given that the spin is a constant of motion, show that the equation of item a supports solutions for which $\theta = \theta_0$ for all times, if $\dot{\phi}_0$ and the spin obey specific conditions. Find these specific conditions and describe the corresponding motion of the top.

c. Suppose one wishes to study the motion of the top in the body frame. Write down the differential equations that govern the rotational motion of the top in this body frame, and derive explicit formulations of the components of the torque vector in that frame. Also describe what a person in the body frame will observe.

Problem 3

Two point masses (labelled 1 and 2 in the figure below), each with a mass m, are connected by means of a stiff, massless rod of length l. Mass point 1, which can only move along the x-axis, is attached to a massless spring with spring constant k. The other end of the spring is anchored to a wall, at a fixed point $x = 0, z = h$. Mass point 2 can move in the $x - z$-plane.

In equilibrium the x-position of the first mass point is $x = x_{1,\text{eq}} = h$ and angle ϕ between the rod and the negative z-axis is $\phi_{\text{eq}} = 0$. Ignore friction.



a. Find the kinetic energy and potential energy of the full system and from these expressions, find the approximate Lagrangian that describes the dynamics of small departures q_1 of x_1 and small departures q_2 of variable $l\phi$ with respect to their equilibrium values.

   Hint: U_{spring} is proportional to $(L - L_{\text{eq}})^2$, with L the actual length of the spring and L_{eq} is its equilibrium length. So find expressions for L and L_{eq} and assume small deviations of L with respect to L_{eq}.

The Lagrangian of item a yields the following equations for the small-amplitude motion (you don't need to show that):

$$2m\ddot{q}_1 + m\ddot{q}_2 = -\frac{1}{2}kq_1\,, \tag{1a}$$

$$m\ddot{q}_1 + m\ddot{q}_2 = -\frac{mg}{l}q_2\,. \tag{1b}$$

b. Derive expressions for the normal frequencies of this system in terms of the model parameters. Assume from now on that $k = 3mg/l$.
   Explain which steps you make in order to obtain your answer.

c. Find the normal modes of the system governed by Eqs. (1a)-(1b) and describe what motion these normal modes represent.

END

Equation sheet Advanced Mechanics
for final exam and retake (version 2024/2025)

A1. Goniometric relations:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \qquad \cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\sin(2\theta) = 2\sin\theta\cos\theta, \qquad \sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

A2. Spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$
$$dxdydz = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$
$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}}$$
$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\hat{\boldsymbol{\theta}} + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\,\sin\theta + 2r\dot{\theta}\dot{\phi}\,\cos\theta)\,\hat{\boldsymbol{\phi}}$$
$$\boldsymbol{\nabla} f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

A3. Cylindrical coordinates ρ, ϕ, z:

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z$$
$$dxdydz = \rho\,d\rho\,d\phi\,dz$$
$$\mathbf{v} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$
$$\mathbf{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\,\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}$$
$$\boldsymbol{\nabla} f = \hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

A4. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

A5. $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$

B1. Formulation for sliding friction: $F_s = \mu_k F_N$

B2. Impulse and rotational impulse due to a kick:

Impulse $\Delta\mathbf{P} = M\Delta\dot{\mathbf{R}}$, $\qquad \Delta N = \Delta P\, l'$

B3. Motion with variable mass:

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} - \mathbf{v}_{ex}\dot{m}$$

with $\mathbf{v}_{ex}$ velocity of dm relative to m.

C1. Systems of particles:

$$\sum_\alpha \mathbf{F}_\alpha = \frac{d\mathbf{P}}{dt}, \qquad \frac{d\mathbf{L}}{dt} = \boldsymbol{\Gamma}$$

C2. Angular momentum vector: $\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha$
where $\mathbf{r}_\alpha = \mathbf{r}_\alpha - \mathbf{R}$, $\quad \mathbf{p}'_\alpha = \mathbf{p}_\alpha - \mathbf{P}$.

C3. Moment of inertia about an arbitrary axis: $I = \int \varrho\, r_\perp^2\, dV$

D1. Lagrange's equations (including constraint forces):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} + \lambda_k \frac{\partial f_k}{\partial q_i}$$

D2. Hamilton's variational principle:

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = 0$$

D3. Hamiltonian function:

$$\mathcal{H} = p_i \dot{q}_i - \mathcal{L}$$

D4. Hamilton's canonical equations:

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

E1. Noninertial reference frames:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\delta_0} = \dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r} + \mathbf{V}$$

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\delta_0} = \ddot{\mathbf{r}} + \dot{\mathbf{\Omega}} \times \mathbf{r} + 2\mathbf{\Omega} \times \dot{\mathbf{r}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + \mathbf{A}$$

F1. Moment of inertia tensor:

$$\mathbf{I} = \sum_\alpha m_\alpha (\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha)\mathbf{1} - \sum_\alpha m_\alpha \mathbf{r}_\alpha \mathbf{r}_\alpha$$

G1. Euler equations: $\Gamma_1 = \lambda_1 \dot{\omega}_1 + \omega_2 \omega_3 (\lambda_3 - \lambda_2)$
(other equations follow by cyclic permutation of indices)

G2. Free precession in body frame: angular frequency $\Omega_b = \left(\frac{\lambda_1 - \lambda_3}{\lambda_1}\right) \omega \cos\alpha$
and $\tan\theta = \frac{\lambda_1}{\lambda_3} \tan\alpha$

G3. Free precession in space frame: angular frequency $\Omega_s = \omega \left[\sin^2\alpha + \frac{\lambda_3^2}{\lambda_1^2} \cos^2\alpha\right]^{1/2}$

H1. Components rotation vector in $x'y'z'$-frame:

$$\omega_{x'} = -\dot{\phi} \sin\theta \qquad \omega_{y'} = \dot{\theta}, \qquad \omega_{z'} = \dot{\phi} \cos\theta + \dot{\psi}$$
