1 a. f_* → reciprocal Coriolis parameter

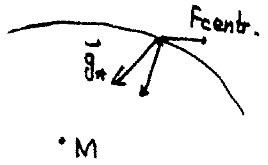
definition : $f_* = 2\Omega \cos\varphi$ where Ω ang. speed of rotation of Earth, φ latitude

Terms $-f_* w$ in x mom. balance and $f_* u$ in z-momentum balance are Coriolis forces/mass related to ~~hor. component~~ component of rotation vector $\vec{\Omega}$ that is tangential to Earth

So in case of motion in x-direction: $f_* u$ causes acceleration in vertical direction. In case of motion in z-direction: acceleration in x direction

b. One term : $-\rho \vec{g}$ in vertical momentum balance.

In fact, $\vec{g}$ is apparent gravity, which involves true gravity $\vec{g}_*$ + centr. force/mass F_{centr}



Note : $\vec{g}_*$ points to centre of Earth
$\vec{g}$ is perpendicular to local surface of Earth (deformed due to centr. force)

c. Boussinesq approximation :

assume $\rho = \rho_0 + \rho'(x,y,z,t)$

where ρ_0 is constant and $\rho' \ll \rho_0$.

Plug into vertical momentum balance and write $p = p_0(z) + p'(x,y,z,t)$

where $\frac{dp_0}{dz} = -\rho_0 g$ and use $\mu = \rho\nu$ (ν is kinematic viscosity)

$$\Rightarrow (\rho_0+\rho')\left[\frac{dw}{dt} - f_* u\right] = \overset{-\frac{\partial p'}{\partial z}}{\cancel{-\frac{dp_0}{dz}}} \cancel{-\rho_0 g} - \rho' g + \rho_0 \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Since $\rho' \ll \rho_0$ it follows

$$\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

d. ~~Substitute~~ Use result of c and set $\rho' = 0$, replace p' by p (dynamic pressure!)

Substitute $u = \langle u \rangle + u'$, $v = \langle v \rangle + v'$, $w = \langle w \rangle + w'$, $p = \langle p \rangle + p'$

~~Note~~ → where $\langle \cdot \rangle$ over turbulence averaged variables
$(\cdot)'$ turbulent fluctuations

Note : p' has a different meaning now

~~Subst.~~ For flow with constant density :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

So

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}(ww) - f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

Substitute, average over turbulence and use that $\langle w' \rangle = \langle u' \rangle = \langle v' \rangle = \langle p' \rangle = 0$

$\Rightarrow$ replace $\langle u \rangle \to u$, $\langle v \rangle \to v$, $\langle w \rangle \to w$, $\langle p \rangle \to p$

and the result is

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}(ww) + \frac{\partial}{\partial x}\langle u'w' \rangle + \frac{\partial}{\partial y}\langle v'w' \rangle + \frac{\partial}{\partial z}\langle w'w' \rangle - f_* u =$$

$$- \frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

~~Finally~~ Here $-\langle u'w' \rangle$, $-\langle v'w' \rangle$, $-\langle w'w' \rangle$ are Reynolds stresse

Finaly, parameterise the gradients of the Reynolds stresses:

$$\frac{\partial}{\partial x}\langle u'w' \rangle = -\frac{\partial}{\partial x}\left(A \frac{\partial}{\partial x}\langle w \rangle\right), \quad \frac{\partial}{\partial y}\langle v'w' \rangle = -\frac{\partial}{\partial y}\left(A \frac{\partial}{\partial y}\langle w \rangle\right), \quad \frac{\partial}{\partial z}\langle w'w' \rangle = -\frac{\partial}{\partial z}\left(\nu_E \frac{\partial w}{\partial z}\right)$$

where A/ν_E horizontal/vertical eddy viscosity coefficients.

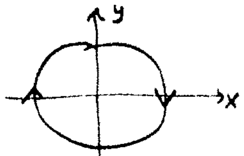
Since $A, \nu_E \gg \nu$ the final result is

$$\boxed{\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(A \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E \frac{\partial w}{\partial z}\right)}$$

e. p is the over turbulence averaged dynamic pressure.

f. Potential temperature of air parcels is conserved when they move through the atmosphere and there is no heat exchange with the surroundings (adiabatic motion).
(More precise: θ is temperature that air parcels acquire if they are moved adiabatically from their vertical level to a fixed reference level)

2 a. The velocity $U_1 > 0$, so circulation is clockwise



topview $\Rightarrow$ rotation axis into the paper

The vortex is anticyclonic, so component of rotation vector $\vec{\Omega}$ along z-axis should be ~~is~~ out of paper. Hence: Northern Hemisphere

b. Rossby number $Ro = \dfrac{U_1}{\Omega R_1}$

Here ~~U~~ $U_1 = 1\ ms^{-1}$, $\Omega = 7.3 \times 10^{-5}\ s^{-1}$, $R_1 = 50\ km$, so $Ro \sim 0.3$

c. Geostrophic balance along e.g. positive x-axis

$$f\, v_\theta = g \frac{d\eta}{dr} \qquad \text{so} \qquad \frac{d\eta}{dr} = -\frac{f}{g} U \frac{r}{R} \qquad r \leq R$$

$$= 0 \qquad r > R$$

Integration :
$$\eta(r) = \frac{f}{2g} U R \left(1 - \frac{r^2}{R^2}\right) \qquad r \leq R$$
$$0 \qquad r > R$$

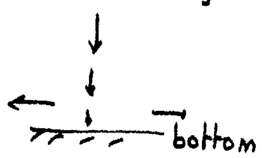
d. Relative circulation $\Gamma = \oint \vec{u} \cdot d\vec{r}$

In this case

$$\Gamma = 2\pi R\, v_\theta (r = R) \qquad \text{so} \qquad \Gamma = -2\pi U_1 R_1$$

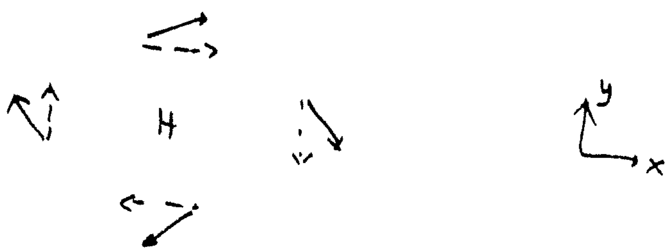
Substitute nrs : $\Gamma \simeq 3.2 \cdot 10^5\ m^2 s^{-1}$

e. The vortex is anticyclonic, i.e., it ~~is~~ has the highest pressure ~~near~~ in the centre.
Thus vertical velocity is downward and vanishes at bottom



In horizontal plane:
radial spreading outward

Thus velocity vectors in horizontal plane (solid : close to bottom ; dashed : in interior) :



2f. Use conservation of mass and conservation of potential vorticity (PV)

(or conservation of absolute circulation)

Conserv. of mass : $\pi R_1^2 h_1 = \pi R_2^2 h_2$

Conserv. of PV : $\dfrac{\zeta_1 + f}{h_1} = \dfrac{\zeta_2 + f}{h_2}$ where $\zeta_1 = -2 \dfrac{U_1}{R_1}$, $\zeta_2 = -2 \dfrac{U_2}{R_2}$

(note factor 2 !)

Development yields : $R_2 = R_1 \left(\dfrac{h_1}{h_2}\right)^{1/2}$

$$U_2 = \left(\frac{h_2}{h_1}\right)^{1/2} U_1 + \frac{1}{2} f R_1 \left(\frac{h_1}{h_2}\right)^{1/2} \left(1 - \frac{h_2}{h_1}\right)$$

Subst. nrs : $R_2 \simeq 61$ km and $U_2 \simeq 1.83$ ms^{-1}