GFD 09 full exam 03-02-2010

1a $\frac{\partial u}{\partial t}$: local acceleration of fluid element in x-direction
$u\frac{\partial u}{\partial x}$: advection of x-component velocity by x-component velocity
$v\frac{\partial u}{\partial y}$: advection of x-component velocity by y-component velocity
$w\frac{\partial u}{\partial z}$: advection of x-component velocity by z-component velocity

The advective terms arise because, when following a fluid element and considering its acceleration $\frac{du}{dt}(x,y,z,t)$, the position of that elements changes in time. So

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\underbrace{\frac{dx}{dt}}_{u} + \frac{\partial u}{\partial y}\underbrace{\frac{dy}{dt}}_{v} + \frac{\partial u}{\partial z}\underbrace{\frac{dz}{dt}}_{w}$$

1b. Boussinesq approximation

$$\rho(x,y,z,t) = \rho_0 + \rho'(x,y,z,t)$$
$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$

where ρ_0 is constant, $\rho' << \rho_0$, p' dynamic pressure and

$$\frac{dp_0}{dz} = -\rho_0 g \qquad (p_0 : \text{static pressure})$$

Apply to zonal momentum equation; result is

$$\rho_0\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + f_* w - fv\right\} = -\frac{\partial p'}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (*)$$

Application of Boussinesq approximation to continuity equation

$$\frac{d\rho}{dt} + \rho\,\vec{\nabla}\cdot\vec{u} = 0 \qquad \text{yields} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \simeq 0$$

Thus in zonal momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) - u\underbrace{\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]}_{\simeq 0} \quad (**)$$

Now, divide (*) by ρ_0, define $\nu = \frac{\mu}{\rho_0}$, replace p' by p and substitute (**) $\Rightarrow$ the desired result.

1c. Reynolds number $Re = \frac{UL}{\nu}$ should exceed a critical value.
Here, U is the horizontal velocity scale, L the horizontal length scale and ν the molecular kinematic viscosity coefficient.
Re measures ratio of inertial (nonlinear) terms and molecular viscous terms in momentum balance.
Yes, geophysical fluids are almost always turbulent (albeit that in many applications the frictional terms can be neglected in describing the mean flow)

1d. Decompose
$u = \langle u \rangle + u'$, similar v, w, p ***
where $\langle u \rangle$ mean and u' turbulent fluctuations; note $\langle u' \rangle = 0$
Substitute decomposition (***) into results of item b
and average equation over turbulent time scale
Finally, replace $\langle u \rangle \to u$, $\langle v \rangle \to v$, $\langle w \rangle \to w$, $\langle p \rangle \to p$
Then

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) + f_* w - f v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$+ \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \frac{\partial}{\partial x}\langle -u'u' \rangle + \frac{\partial}{\partial y}\langle -u'v' \rangle + \frac{\partial}{\partial z}\langle -u'w' \rangle$$

Here, $-\langle u'u' \rangle$, $\langle -u'v' \rangle$, $\langle -u'w' \rangle$ are Reynolds stresses

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2a. Geostrophic balance

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad , \quad fv = \frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

Here, $v = 0$, so $\frac{dp}{dy} = -\rho_0 f u$
$u = u(y)$

$$\Rightarrow \quad p(y) - \hat{p} = -\rho_0 f \int_0^y u(y')\,dy'$$

In domain $|y| < L$: $\quad p(y) = \hat{p} + \frac{\rho_0 f U}{L}\int_0^y y'\,dy' = \hat{p} + \frac{\rho_0 f U}{2L} y^2$

In domain $y > L$: $\quad p(y) = p(L) + \rho_0 f U \int_L^y dy' = \hat{p} + \frac{1}{2}\rho_0 f U L + \rho_0 f U(y-L)$

In domain $y < L$: $\quad p(y) = p(-L) - \rho_0 f U \int_L^y dy' = \hat{p} + \frac{1}{2}\rho_0 f U L - \rho_0 f U(y+L)$

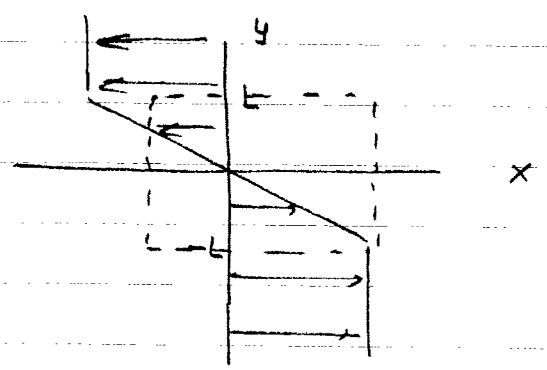
b. Absolute vorticity

$$\zeta_a = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f$$

Here $v = 0$, so

$$\zeta_a = \begin{cases} f & \text{if } y > L \\ \frac{U}{L} + f & \text{if } |y| < L \\ f & \text{if } y < L \end{cases}$$

c.



arrows: velocity
dashed lines: edges of the box

Relative circulation $\Gamma = \iint \zeta \, dS = 4L^2 \frac{U}{L} > 0$

Positive circulation can also be inferred from arrows

Note $\Gamma > 0$ whereas $f < 0$ (Southern Hemisphere)

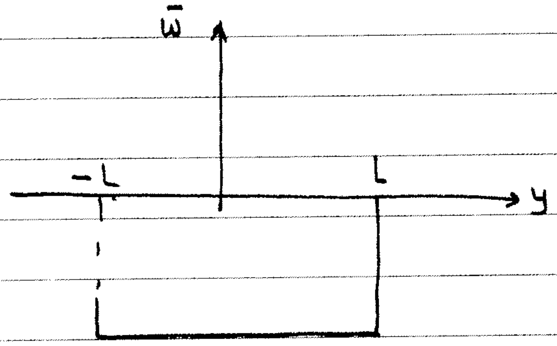
So relative circulation is anticyclonic

2d. Ekman pump on Southern Hemisphere

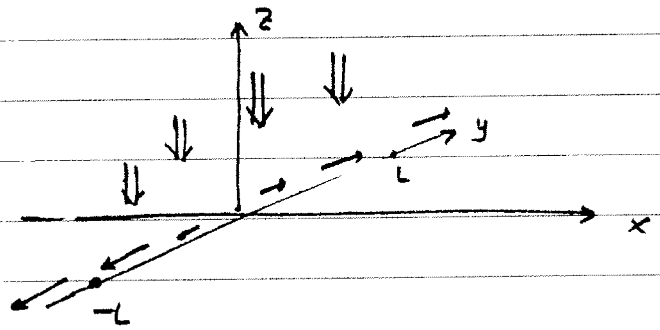
$$\bar{w} = -\frac{d}{2}\bar{\zeta}$$

In this case $\bar{\zeta} = -\frac{du}{dy} = \begin{cases} 0 & y > L \\ U/L > 0 & |y| < L \\ 0 & y < L \end{cases}$

So



In area where $\bar{w} < 0$:



Note cross-isobaric flow (in y-direction) directed away from high pressure core in $y = 0$. Magnitude of this flow increases from $y = 0$ to $|y| = L$, for $|y| > L$ constant

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For answers problems 3 + 4
See GFD09 ~~mid~~exam block 2 (03-02-2010) problems 1 + 2