FULL EXAM GEOPHYSICAL FLUID DYNAMICS

14 March 2012, 9.00 - 12.00 (3 hours)

Four problems (all items have equal weight)

Remark 1: answers may be written in English or Dutch.

Remark 2: in all questions you may use $g = 10$ ms^{-2}, $a = 6400$ km and $\Omega = 7.3 \times 10^{-5}$ s^{-1}.

Problem 1

One of the equations of motion that governs fluid dynamics at molecular scales is the density equation

$$\frac{d\rho}{dt} = 0 . \qquad (*)$$

a. What physical law(s) lead to this equation? No derivations are asked.
Also, give a physical interpretation of this equation.

b. Apply a generalised Boussinesq approximation to the density equation, i.e., substitute

$$\rho = \rho_0 + \tilde{\rho}(z) + \rho'(x, y, z, t)$$

into equation (*) and show that the result is

$$\frac{d\rho'}{dt} + w \frac{d\tilde{\rho}}{dz} = 0 .$$

Specify assumptions (if needed) to arrive at this quation.

c. Apply a Reynolds averaging procedure to the equation of item b. and show that the final result can be written as

$$\frac{d\rho}{dt} + w \frac{d\tilde{\rho}}{dz} = \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial \rho}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial \rho}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right) .$$

What is the meaning of variable ρ and parameters $\mathcal{A}$, κ_E in this result?

d. Consider a fluid that is characterised by

$$\rho = \hat{\rho}(x) , \qquad u = U , \qquad v = 0 , \qquad w = 0 ,$$

with U a constant.
Using the result of item c., derive the equation for $\hat{\rho}(x)$ and explain its physical meaning.

For problem 2: P.T.O.

Problem 2

In a homogenous ocean (depth H=3 km) a zonal geostrophic flow is present that has a profile

$$\bar{u} = U \exp\left(-(y/L)^2\right) ,$$

with $U = 1 \text{ m s}^{-1}$ and $L = 100$ km. Assume that $\nu_E = 10^{-2} \text{ m}^2 \text{ s}^{-1}$ and $f = 10^{-4} \text{ s}^{-1}$.

a. Compute the Rossby number and vertical Ekman number at location $y = 0$.

b. Write down the equations of motion and boundary conditions that govern the flow in the bottom Ekman layer (no derivation asked).

c. Sketch the zonal transport U and meridional transport V, which are due to the flow in the bottom Ekman layer, as a function of the south-north coordinate y.
Explain your answer.

d. Compute the maximum geostrophic vertical velocity $\bar{w}$ that is produced by the bottom Ekman layer.

For problem 3: next page

Problem 3

Quasi-geostrophic flow in the atmosphere is governed by the equations

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = 0\,, \qquad q_1 = \nabla^2\psi_1 + \frac{1}{2R^2}(\psi_2 - \psi_1) + f_0 + \beta_0\, y\,,$$
$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = 0\,, \qquad q_2 = \nabla^2\psi_2 - \frac{1}{2R^2}(\psi_2 - \psi_1) + f_0 + \beta_0\, y\,.$$

with $R = (g'H)^{1/2}/|f_0|$ and H the depth of the atmosphere.

a. Name the variables q_1, q_2 and name the parameters R, f_0 and β_0.
Also, describe the physical meaning of the differential equation for q_1 and the physical meaning of the three terms that occur in the definition of q_1.

b. It is convenient to derive and analyse equations for the two variables

$$\psi_T = \frac{1}{2}(\psi_1 + \psi_2)\,, \qquad \psi_B = \frac{1}{2}(\psi_1 - \psi_2)\,.$$

Show that the linearised equation for variable ψ_B reads

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_B - \frac{1}{R^2}\psi_B\right) + \beta_0\frac{\partial \psi_B}{\partial x} = 0\,.$$

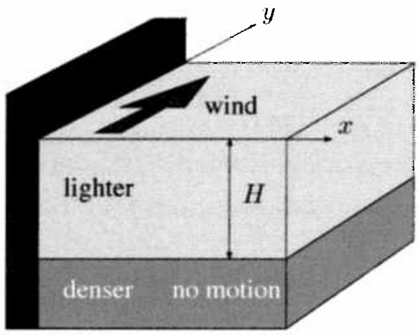
c. Consider the equation for variable ψ_B that is given in item b.
Substitute wave-like solutions in this equation for ψ_B and derive the dispersion relation of these waves.
What is the name of these waves?

d. Under certain conditions the two-layer quasi-gesotrophic model describes wave-like solutions of which the amplitude grows exponentially in time.
What conditions are necessary to find such solutions?
Also, name the underlying physical mechanism and give an estimate of the zonal wavelength of the fastest growing wave.

For problem 4: P.T.O.

Problem 4

Consider a two-layer ocean, of which the lower layer is infinitely thick. This ocean is bounded by a coast at $x = 0$. Initially, the system is at rest and the depth of the upper layer has a constant and spatially uniform value H. At time $t = 0$, a spatially uniform wind starts to blow along the coast (see figure), which ceases at time $t = t_*$.



Initially

The dynamics are governed by the nonlinear shallow water equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}, \qquad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = \frac{\tau}{\rho_0 h},$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0.$$

where the wind stress is

$$\tau^y = \begin{cases} \hat{\tau}(t) & \text{for } 0 \le t \le t_*, \\ 0 & \text{for } t > t_*. \end{cases}$$

Below, the adjusted, steady end state of the system (attained for $t \to \infty$) is analysed.

a. Show that the end state is characterised by $u = 0$.

b. Use the result of item a, as well as conservation of potential vorticity, to derive two differential equations for v and h of the end state.
Present these equations and show that their solutions are

$$h = H + Ae^{-x/R}, \qquad v = -A\left(\frac{g'}{H}\right)^{1/2} e^{-x/R},$$

where A is an integration constant.

c. Consider the case that outcropping of the lower layer occurs at $x = d$. This means that in the region $0 \le x \le d$ dense, cold bottom water reaches the surface.
Express A in terms of d and other model parameters and sketch the interface of the adjusted state.

d. An expression for distance d is found from integratation of the longshore momentum equation, yielding

$$d = -R + \int_0^{t_*} \frac{\hat{\tau}}{\rho_0 f h}\, dt.$$

Give a physical interpretation of this result.

END