EXAM GEOPHYSICAL FLUID DYNAMICS

4 November 2013, 9.00 - 12.00 (3 hours)

Four problems (all items have equal weight)

Remark 1: answers may be written in English or Dutch.

Remark 2: in all questions you may use $g = 10$ ms^{-2}, $a = 6400$ km and $\Omega = 7.3 \times 10^{-5}$ s^{-1}.

Problem 1

Consider the continuity equation and momentum equations for a molecular viscous fluid on a rotating Earth, as are given on the supplementary equation sheet.

a. Are these equations exact?
If not, what simplifications have been made?

b. Specify the Boussinesq approximation and use it to derive a simpler version of the mass budget.

c. Derive for a fluid with constant density its momentum budget in the south-north direction that results from application of the Reynolds averaging procedure.
What are the similarities and differences with respect to the momentum budget for a molecular viscous fluid?

d. Write down the hydrostatic balance, discuss its physical meaning and specify the condition(s) for which this balance is valid.
Are there additional terms in the continuity and momentum equations that can be ignored in case that the flow is hydrostatic? If so, which one(s)?

For problem 2: P.T.O.

Problem 2

In the area south of the equator the wind stress over the surface of the ocean is given by

$$\tau^x = -T \cos\left(\frac{2\pi y}{W}\right), \qquad \tau^y = 0.$$

Here, $0 \le y \le W$ and T is a positive constant.
This winds stress generates steady and linear large-scale flow (components u and v) that is uniform in the zonal direction. In the interior of the ocean the flow is geostrophic (components $\bar{u}$ and $\bar{v}$). Furthermore, assume parameters f and ν_E to be constants.

a. Sketch the velocity vector (with components $u - \bar{u}$ and $v - \bar{v}$) in the surface layer of the ocean as a function of vertical coordinate z at location $y = 0$.
Explain your answer.

b. Compute and sketch the distribution of the surface Ekman pumping velocity as a function of y.
Also, give a short interpretation of your result.

c. Assuming that the Ekman pumping velocity near the bottom equals the surface Ekman pumping velocity, derive an algebraic expression for the zonal geostrophic velocity $\bar{u}$ that explicitly contains parameter T.

d. Derive an algebraic expression for the absolute circulation in the interior of the ocean in the domain defined by $0 \le x \le L$ and $0 \le y \le W/2$. Here, L is a fixed length.

For problem 3: next page

Problem 3

Consider the following equation for free surface elevations:

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t} \nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0, \qquad (*)$$

a. Substitute wave-like solutions in Equation (*) above and show that the dispersion relation of the waves can be recapitulated as

$$\left(k_x + \frac{\beta_0}{2\omega}\right)^2 + k_y^2 = \frac{\beta_0^2}{4\omega^2} - \frac{1}{R^2}.$$

What is the name of these waves?

b. Consider waves for which $\omega = \frac{1}{2}\omega_{max}$, where ω_{max} is the maximum frequency that the waves governed by Equation (*) can attend.
Derive an expression for ω in terms of the parameters R and β_0.

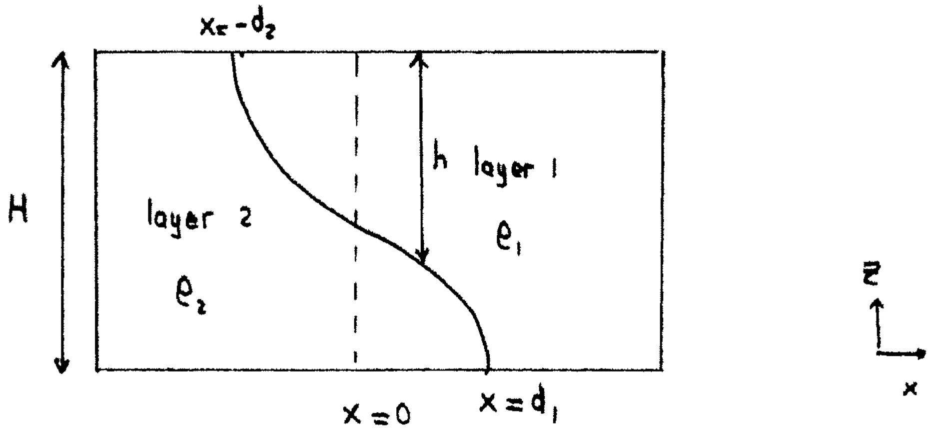
c. The waves of item b ($\omega = \frac{1}{2}\omega_{max}$) also have a meridional wavenumber $k_y = \sqrt{2}/R$ and their energy propagation speed has an eastward component.
Compute the zonal wavenumber k_x of these waves.

d. Sketch, at a fixed time, the crest and trough lines of the waves of item c in the $x-y$ plane, Northern Hemisphere.
Also indicate, in the same figure, with arrows the velocity experienced by fluid particles. Explain your answer.

For problem 4: P.T.O.

Problem 4

Two layers of fluids, with densities ρ_1 and ρ_2, are initially separated by a vertical wall at $x = 0$ (the dashed line in the figure below). In the y-direction you can assume uniform conditions.



At time $t = 0$ the vertical wall is removed and the system adjusts to a final steady state. In the figure, the interface between the two fluids in the end state is indicated by the solid curve (from $x = -d_2$ to $x = d_1$). Note that thickness $h = H - a$. The dynamics is governed by the nonlinear, frictionless shallow water equations for a two-layer system on the f-plane.

a. Considering the situation sketch above, is density ρ_1 larger or smaller than ρ_2? Motivate your answer.

b. From analysing the equations of motion, it follows that

$$\frac{dv_1}{dx} = f\left[\frac{h}{H} - 1\right], \qquad \frac{dv_2}{dx} = -f\frac{h}{H},$$

where v_1, v_2 are the velocities in the layers with densities ρ_1, ρ_2.
Describe briefly how these results are obtained, and define the principle that is crucial in this respect.

c. A third equation relating v_1, v_2 and h in the end state is

$$f(v_1 - v_2) = g'\frac{dh}{dx}.$$

Name this balance, and explain how it is derived from the given equations of motion.

d. From the equations of items b and c the thickness $h(x)$ of the end state can be found, using the boundary conditions $h(x = -d_2) = 0$ and $h(x = d_1) = H$. In order to determine the locations d_1 and d_2 two additional constraints are needed.
Describe these constraints, both physically and in terms of mathematical expressions.

END

GFD 2013 Equation sheet

Continuity and momentum equations: molecular viscous fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\begin{aligned}
\rho\left(\frac{du}{dt} + f_* w - f v\right) &= -\frac{\partial p}{\partial x} & &+ \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
\rho\left(\frac{dv}{dt} + f u\right) &= -\frac{\partial p}{\partial y} & &+ \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\\
\rho\left(\frac{dw}{dt} - f_* u\right) &= -\frac{\partial p}{\partial z} - \rho g & &+ \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}$$

Relative circulation and relative vorticity

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS$$

where S is the surface enclosed by contour C.

Shallow water equations

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f v &= -g\frac{\partial \eta}{\partial x}\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f u &= -g\frac{\partial \eta}{\partial y}\\
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0
\end{aligned}$$

Ekman pump (Northern Hemisphere)

$$\overline{w} = \frac{d}{2}\overline{\zeta}, \qquad d = \left(\frac{2\nu_E}{f}\right)^{1/2} \qquad \text{and} \qquad w_{\mathrm{E}k} = \frac{1}{\rho_0 f}\left[\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right]$$

Linear shallow water equations for 2-layer model

$$\frac{\partial u_1}{\partial t} - f\,v_1 = -g\frac{\partial \eta}{\partial x}, \qquad \frac{\partial v_1}{\partial t} + f\,u_1 = -g\frac{\partial \eta}{\partial y},$$

$$\frac{\partial u_2}{\partial t} - f\,v_2 = -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x}, \qquad \frac{\partial v_2}{\partial t} + f\,u_2 = -g\frac{\partial \eta}{\partial y} - g'\frac{\partial a}{\partial y},$$

$$\frac{\partial}{\partial t}(\eta - a) + \frac{\partial}{\partial x}(H_1 u_1) + \frac{\partial}{\partial y}(H_1 v_1) = 0,$$

$$\frac{\partial a}{\partial t} + \frac{\partial}{\partial x}((H_2 - b)u_2) + \frac{\partial}{\partial y}((H_2 - b)v_2) = 0.$$

Characteristic depth: $\bar{h} = H_1\,H_2/(H_1 + H_2)$.

Generalised equation for barotropic planetary/topographic waves

$$\frac{\partial \eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta - \beta_0\,R^2\frac{\partial \eta}{\partial x} + \frac{\alpha_0 g}{f_0}\frac{\partial \eta}{\partial x} = 0.$$

Complex variable

$\phi \equiv |\phi|\,e^{i\theta} \equiv \phi_r + i\phi_i$, where $|\phi|^2 = \phi_r^2 + \phi_i^2$, $\tan\theta = \phi_i/\phi_r$.

QG Theory for a continuously stratified fluid

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0}J(p', \nabla^2 p') + \beta_0\frac{\partial p'}{\partial x}\right],$$

$$\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0}J(p', \rho') - \frac{\rho_0 N^2}{g}w = 0, \qquad N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz},$$

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \qquad q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y.$$

Two-layer QG model

$$q_1 = \nabla^2\psi_1 + \frac{1}{2R^2}(\psi_2 - \psi_1) + f_0 + \beta_0\,y,$$

$$q_2 = \nabla^2\psi_2 - \frac{1}{2R^2}(\psi_2 - \psi_1) + f_0 + \beta_0\,y, \qquad R = \frac{\sqrt{g'H}}{2f_0},$$

$$w = \frac{2f_0}{N^2 H}\left[\frac{\partial}{\partial t}(\psi_2 - \psi_1) + J(\psi_1, \psi_2)\right], \qquad N^2 = \frac{2g'}{H},$$

$$a = \frac{f_0}{g}(\psi_2 - \psi_1), \qquad \rho' = \rho_0\frac{f_0}{2gH}(\psi_2 - \psi_1).$$