FINAL EXAM GEOPHYSICAL FLUID DYNAMICS

2 November 2015, 9.00 - 11.00 (2 hours)

Two problems (all items have equal weight)

Remark 1: answers may be written in English or Dutch.

Remark 2: in all questions you may use $g = 10$ ms^{-2}, $a = 6400$ km and $\Omega = 7.3 \times 10^{-5}$ s^{-1}.

Problem 1

Quasi-geostrophic flow in the atmosphere is governed by the equations

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = 0\,, \qquad q_1 = \nabla^2\psi_1 + \frac{1}{2R^2}(\psi_2 - \psi_1) + \beta_0\, y\,,$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = 0\,, \qquad q_2 = \nabla^2\psi_2 - \frac{1}{2R^2}(\psi_2 - \psi_1) + \beta_0\, y\,.$$

with $R = (g'H)^{1/2}/|f_0|$ and H the depth of the atmosphere.

a. Name the variables q_1, q_2 and name the parameters R, f_0 and β_0.
Also, describe the physical meaning of the differential equation for q_1 and the physical meaning of the three terms that occur in the definition of q_1.

b. It is convenient to derive and analyse equations for the two variables

$$\psi_T = \frac{1}{2}(\psi_1 + \psi_2)\,, \qquad \psi_B = \frac{1}{2}(\psi_1 - \psi_2)\,.$$

Show that the linearised equation for variable ψ_B reads

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_B - \frac{1}{R^2}\psi_B\right) + \beta_0 \frac{\partial \psi_B}{\partial x} = 0\,.$$

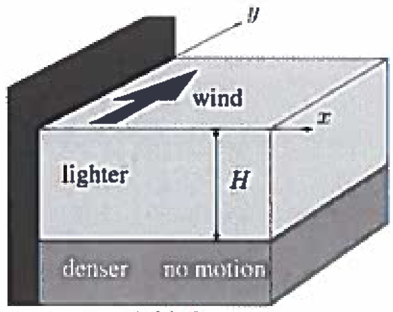
c. Consider the equation for variable ψ_B that is given in item b.
Substitute wave-like solutions in this equation for ψ_B and derive the dispersion relation of these waves.
What is the name of these waves?

d. Under certain conditions the two-layer quasi-gesotrophic model describes wave-like solutions of which the amplitude grows exponentially in time.
What conditions are necessary to find such solutions?
Also, name the underlying physical mechanism and give a (rough) estimate of the zonal wavelength of the fastest growing wave.

For problem 2: P.T.O.

Problem 2

Consider a two-layer ocean, of which the lower layer is infinitely thick. This ocean is bounded by a coast at $x = 0$. Initially, the system is at rest and the depth of the upper layer has a constant and spatially uniform value H. At time $t = 0$, a spatially uniform wind starts to blow along the coast (see figure), which ceases at time $t = t_*$.



Initially

The dynamics are governed by the nonlinear shallow water equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}, \qquad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = \frac{\tau}{\rho_0 h},$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0.$$

where the wind stress is

$$\tau^y = \begin{cases} \hat{\tau}(t) & \text{for } 0 \le t \le t_*, \\ 0 & \text{for } t > t_*. \end{cases}$$

Below, the adjusted, steady end state of the system (attained for $t \to \infty$) is analysed.

a. Show that the end state is characterised by $u = 0$.

b. Use the result of item a, as well as conservation of potential vorticity, to derive two differential equations for v and h of the end state.
Present these equations and show that their solutions are

$$h = H + A\,e^{-x/R}, \qquad v = -A\left(\frac{g'}{H}\right)^{1/2} e^{-x/R},$$

where A is an integration constant.

c. Consider the case that outcropping of the lower layer occurs at $x = d$. This means that in the region $0 \le x \le d$ dense, cold bottom water reaches the surface.
Express A in terms of d and other model parameters and sketch the interface of the adjusted state.

d. An expression for distance d is found from integration of the longshore momentum equation, yielding

$$d = -R + \int_0^{t_*} \frac{\hat{\tau}}{\rho_0 f h}\,dt.$$

Give a physical interpretation of this result.

END

GFD 2015 Equation sheet

Continuity and momentum equations: molecular viscous fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w) = 0$$

$$\begin{aligned}
\rho\left(\frac{du}{dt} + f_* w - f\,v\right) &= -\frac{\partial p}{\partial x} & &+\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
\rho\left(\frac{dv}{dt} + f\,u\right) &= -\frac{\partial p}{\partial y} & &+\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\\
\rho\left(\frac{dw}{dt} - f_* u\right) &= -\frac{\partial p}{\partial z} - \rho g & &+\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}$$

Relative circulation and relative vorticity

$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{u})\cdot \mathbf{n}\, dS$$

where S is the surface enclosed by contour C.

Shallow water equations

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f\,v &= -g\frac{\partial h}{\partial x}\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f\,u &= -g\frac{\partial h}{\partial y}\\
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0
\end{aligned}$$

Ekman pump (Northern Hemisphere)

$$\overline{w} = \frac{d}{2}\overline{\zeta}, \qquad d = \left(\frac{2\nu_E}{f}\right)^{1/2} \qquad \text{and} \qquad w_{Ek} = \frac{1}{\rho_0 f}\left[\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right]$$

Linear shallow water equations for 2-layer model

$$\frac{\partial u_1}{\partial t} - f\,v_1 = -g\frac{\partial \eta}{\partial x}, \qquad \frac{\partial v_1}{\partial t} + f\,u_1 = -g\frac{\partial \eta}{\partial y},$$

$$\frac{\partial u_2}{\partial t} - f\,v_2 = -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x}, \qquad \frac{\partial v_2}{\partial t} + f\,u_2 = -g\frac{\partial \eta}{\partial y} - g'\frac{\partial a}{\partial y},$$

$$\frac{\partial}{\partial t}(\eta - a) + \frac{\partial}{\partial x}(H_1 u_1) + \frac{\partial}{\partial y}(H_1 v_1) = 0,$$

$$\frac{\partial a}{\partial t} + \frac{\partial}{\partial x}((H_2 - b)u_2) + \frac{\partial}{\partial y}((H_2 - b)v_2) = 0.$$

Characteristic depth: $\bar{h} = H_1\,H_2/(H_1 + H_2)$.

Generalised equation for barotropic planetary/topographic waves

$$\frac{\partial \eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta - \beta_0\,R^2\frac{\partial \eta}{\partial x} + \frac{\alpha_0 g}{f_0}\frac{\partial \eta}{\partial x} = 0.$$

Complex variable

$\phi \equiv |\phi|\,e^{i\theta} \equiv \phi_r + i\phi_i$, where $|\phi|^2 = \phi_r^2 + \phi_i^2$, $\tan\theta = \phi_i/\phi_r$.

QG Theory for a continuously stratified fluid

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0}J(p', \nabla^2 p') + \beta_0\frac{\partial p'}{\partial x}\right],$$

$$\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0}J(p', \rho') - \frac{\rho_0 N^2}{g}w = 0, \qquad N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz},$$

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \qquad q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y.$$

Two-layer QG model

$$q_1 = \nabla^2\psi_1 + \frac{1}{2R^2}(\psi_2 - \psi_1) + \beta_0\,y,$$

$$q_2 = \nabla^2\psi_2 - \frac{1}{2R^2}(\psi_2 - \psi_1) + \beta_0\,y, \qquad R = \frac{\sqrt{g'H}}{2f_0},$$

$$w = \frac{2f_0}{N^2 H}\left[\frac{\partial}{\partial t}(\psi_2 - \psi_1) + J(\psi_1, \psi_2)\right], \qquad N^2 = \frac{2g'}{H},$$

$$a = \frac{f_0}{g}(\psi_2 - \psi_1), \qquad \rho' = \rho_0\frac{f_0}{2gH}(\psi_2 - \psi_1).$$