FINAL EXAM GEOPHYSICAL FLUID DYNAMICS

7 November 2016, 9:00 - 11.00 (2 hours)

Two problems, total 45 points.

Please write your answers for each problem on a separate sheet of paper and put your name and student number on each sheet.

Answers may be written in English or Dutch. Please write clearly and not with a pencil! Remember to write down the correct units of the quantities/numbers you calculate.

In all questions you may use $g = 10$ ms^{-2}, $r_a = 6400$ km and $\Omega = 7.3 \times 10^{-5}$ s^{-1}.

Problem 1

Consider the following equation for free surface elevations:

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t} \nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0 \,, \qquad (*)$$

a. Name and define the parameters R and β_0.
Also, explain the physical meaning of these parameters. (5 points)

b. Substitute wave-like solutions in Equation (*) above and show that the dispersion relation of the waves can be recapitulated as

$$\left(k_x + \frac{\beta_0}{2\omega}\right)^2 + k_y^2 = \frac{\beta_0^2}{4\omega^2} - \frac{1}{R^2} \,.$$

What is the name of these waves? (5 points)

c. Consider waves for which $\omega = \frac{1}{2}\omega_{max}$, where ω_{max} is the maximum frequency that the waves governed by Equation (*) can attend.
Derive an expression for ω in terms of the parameters R and β_0. (5 points)

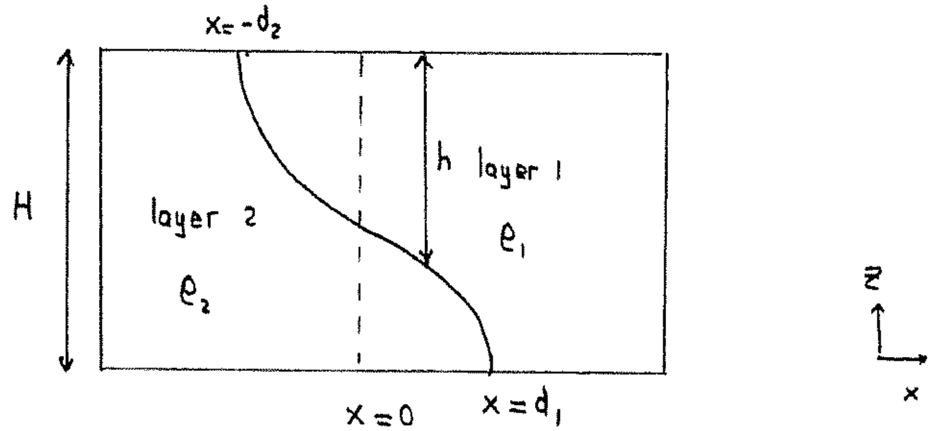
d. The waves of item b ($\omega = \frac{1}{2}\omega_{max}$) also have a meridional wavenumber $k_y = \sqrt{2}/R$ and their energy propagation speed has an eastward component.
Compute the zonal wavenumber k_x of these waves. You will find two solutions for k_x, please argue which one is the one we are looking for here. (5 points)

e. Sketch, at a fixed time, the crest and trough lines of the waves of item c in the $x - y$ plane, Northern Hemisphere.
Also indicate, in the same figure, with arrows the velocity experienced by fluid particles. Explain your answer. (5 points)

For problem 2: P.T.O.

Problem 2

Two layers of fluids, with densities ρ_1 and ρ_2, are initially separated by a vertical wall at $x = 0$ (the dashed line in the figure below). In the y-direction you can assume uniform conditions.



At time $t = 0$ the vertical wall is removed and the system adjusts to a final steady state. In the figure, the interface between the two fluids in the end state is indicated by the solid curve (from $x = -d_2$ to $x = d_1$). Note that thickness $h = H - a$. The dynamics is governed by the nonlinear, frictionless shallow water equations for a two-layer system on the f-plane.

a. Considering the situation sketch above, is density ρ_1 larger or smaller than ρ_2? Motivate your answer. (5 points)

b. By analysing the equations of motion, it follows that

$$\frac{dv_1}{dx} = f\left[\frac{h}{H} - 1\right], \qquad \frac{dv_2}{dx} = -f\frac{h}{H},$$

where v_1, v_2 are the velocities in the layers with densities ρ_1, ρ_2.
Derive this result and explain the principle that is crucial in this respect. (5 points)

c. A third equation relating v_1, v_2 and h in the end state is

$$f\,(v_1 - v_2) = g'\frac{dh}{dx}.$$

Name this balance, and derive it from the given equations of motion (explain your steps). (5 points)

d. From the equations of items b and c the thickness $h(x)$ of the end state can be found, using the boundary conditions $h(x = -d_2) = 0$ and $h(x = d_1) = H$. In order to determine the locations d_1 and d_2 two additional constraints are needed.
Describe these constraints, both physically and in terms of mathematical expressions. (5 points)

END

GFD 2016 Equation sheet

Continuity and momentum equations: molecular viscous fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\rho\left(\frac{du}{dt} + f_* w - f v\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{dv}{dt} + f u\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Relative circulation and relative vorticity

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS$$

where S is the surface enclosed by contour C.

Shallow water equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

Ekman pump (Northern Hemisphere)

$$\overline{w} = \frac{d}{2}\overline{\zeta}, \qquad d = \left(\frac{2\nu_E}{f}\right)^{1/2} \qquad \text{and} \qquad w_{\mathrm{E}k} = \frac{1}{\rho_0 f}\left[\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right]$$

Linear shallow water equations for 2-layer model

$$\frac{\partial u_1}{\partial t} - f\,v_1 = -g\frac{\partial \eta}{\partial x}\,, \qquad \frac{\partial v_1}{\partial t} + f\,u_1 = -g\frac{\partial \eta}{\partial y}\,,$$

$$\frac{\partial u_2}{\partial t} - f\,v_2 = -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x}\,, \qquad \frac{\partial v_2}{\partial t} + f\,u_2 = -g\frac{\partial \eta}{\partial y} - g'\frac{\partial a}{\partial y}\,,$$

$$\frac{\partial}{\partial t}(\eta - a) + \frac{\partial}{\partial x}(H_1 u_1) + \frac{\partial}{\partial y}(H_1 v_1) = 0\,,$$

$$\frac{\partial a}{\partial t} + \frac{\partial}{\partial x}((H_2 - b)u_2) + \frac{\partial}{\partial y}((H_2 - b)v_2) = 0\,.$$

Characteristic depth: $\bar{h} = H_1\,H_2/(H_1 + H_2)$.

Generalised equation for barotropic planetary/topographic waves

$$\frac{\partial \eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta - \beta_0\,R^2\frac{\partial \eta}{\partial x} + \frac{\alpha_0 g}{f_0}\frac{\partial \eta}{\partial x} = 0\,.$$

Complex variable

$$\phi \equiv |\phi|\,e^{i\theta} \equiv \phi_r + i\phi_i, \quad \text{where } |\phi|^2 = \phi_r^2 + \phi_i^2,\ \tan\theta = \phi_i/\phi_r.$$

QG Theory for a continuously stratified fluid

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0}J(p', \nabla^2 p') + \beta_0\frac{\partial p'}{\partial x}\right]\,,$$

$$\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0}J(p', \rho') - \frac{\rho_0 N^2}{g}w = 0\,, \qquad N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}\,,$$

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0\,, \qquad q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y\,.$$

Two-layer QG model

$$q_1 = \nabla^2\psi_1 + \frac{1}{2R^2}(\psi_2 - \psi_1) + f_0 + \beta_0\,y\,,$$

$$q_2 = \nabla^2\psi_2 - \frac{1}{2R^2}(\psi_2 - \psi_1) + f_0 + \beta_0\,y\,, \qquad R = \frac{\sqrt{g'H}}{2f_0}\,,$$

$$w = \frac{2f_0}{N^2 H}\left[\frac{\partial}{\partial t}(\psi_2 - \psi_1) + J(\psi_1, \psi_2)\right]\,, \qquad N^2 = \frac{2g'}{H}\,,$$

$$a = \frac{f_0}{g}(\psi_2 - \psi_1)\,, \qquad \rho' = \rho_0\frac{f_0}{2gH}(\psi_2 - \psi_1)\,.$$