EXAM GEOPHYSICAL FLUID DYNAMICS

4 January 2019, 9.00 - 12.00 hours

Four problems (all items have equal weight)

Remark : in all questions you may use $g = 10$ ms^{-2}, $a = 6400$ km and $\Omega = 7.3 \times 10^{-5}$ s^{-1}.

Problem 1

The equations of motion that govern the dynamics of a molecular viscous fluid contain the two equations

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0\,, \qquad\qquad \frac{dq}{dt} = \kappa_q\,\nabla^2 q\,.$$

a. What physical law(s) lead to these two equations? No derivations are asked. Also, describe the meaning of variables ρ and q and of parameter κ_q.

b. Specify the Boussinesq approximation and apply it to the first equation given above. What is the final result?

c. Apply a Reynolds averaging procedure to the second equation given above and show that the final result can be written as

$$\frac{dq}{dt} = \frac{\partial}{\partial x}\left(\mathcal{A}\frac{\partial q}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathcal{A}\frac{\partial q}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial q}{\partial z}\right)\,.$$

What is the meaning of variable q and of parameters $\mathcal{A}, \kappa_E$ in this result?

d. Consider a fluid that is characterised by a field

$$q = \hat{q}(x) + \tilde{q}(z)\,, \qquad \hat{q}(x) = \beta\,x\,, \qquad \tilde{q}(z) = Q\left(1 + \cos\left(\frac{\pi z}{H}\right)\right)$$

and a velocity field

$$u = U(z)\,, \qquad v = 0\,, \qquad w = 0\,.$$

Here, β, Q and H are positive constants and $0 \leq z \leq H$.
Use the result of item c, with $\mathcal{A} = 0$ and κ_E constant, to find $U(z)$.

For problem 2: P.T.O.

Problem 2

A cyclonic storm in the atmosphere imposes the following wind stress on the ocean:

$$\tau^x = -A\, y\, e^{-(x^2+y^2)/\lambda^2}\,, \qquad \tau^y = A\, x\, e^{-(x^2+y^2)/\lambda^2}\,,$$

where A and λ are positive constants and x, y are coordinates on a local f-plane.

a. Is this storm located on the Northern Hemisphere or Southern Hemisphere?
   Motivate your answer.

b. Present the two momentum equations and corresponding boundary conditions that govern the flow in the surface Ekman layer of the ocean.
   Explain which terms are retained and which terms are neglected.

c. Compute the distribution of the Ekman pumping velocity in the ocean that is induced by the wind.
   Also, give a physical interpretation of your result.

For problem 3: next page

Problem 3

Consider a fluid that is confined between two walls, located at $x = -L$ and $x = L$. The fluid consists of two layers, of which the lower layer is motionless. The dynamics are governed by

$$\frac{du}{dt} - f\,v = -g'\frac{\partial h}{\partial x}\,, \qquad \frac{dv}{dt} + f\,u = -g'\frac{\partial h}{\partial y}\,,$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0\,.$$

a. Show that the equations of motion allow for a steady state, in which conditions are uniform in the y-direction, $u = 0$ and v obeys the geostrophic balance.

At time $t = 0$ the fluid state is $h = H$, $u = a\,x(x + L)(x - L)$ and $v = b\,x$, where H, a and b are constants.

b. Assuming longshore uniform conditions, show that at any time t, the following relationship holds:

$$\frac{\partial v}{\partial x} = \alpha h + \beta\,,$$

where α and β are constants.
Discuss what principle you use, and express α and β in terms of the model parameters.

c. Now assume that in the expression of item b, $v = v_e$ and $h = h_e$, where v_e and h_e are the y-component of the velocity and layer thickness in the state that is described in item a. Show that the differential equation governing h_e reads

$$\frac{d^2 h_e}{dx^2} = \mu^2\,h_e + C\,.$$

Determine the constants μ and C and explain how you obtain your answer.

d. Find the general solution for h_e and specify the conditions that determine the integration constants.

For problem 4: P.T.O.

Problem 4

Quasi-geostrophic flow in the atmosphere is governed by the equations

$$\frac{dq_1}{dt} = 0\,, \qquad q_1 = \nabla^2\psi_1 - \frac{1}{2R^2}(\psi_1 - \psi_2) + f_0 + \beta_0\, y\,,$$
$$\frac{dq_2}{dt} = 0\,, \qquad q_2 = \nabla^2\psi_1 + \frac{1}{2R^2}(\psi_1 - \psi_2) + f_0 + \beta_0\, y\,.$$

a. Name and describe the physical meaning of variables q_1, q_2 and of parameters R, β_0.

b. It is convenient to derive and analyse equations for the two variables

$$\psi_T = \frac{1}{2}(\psi_1 + \psi_2)\,, \qquad \psi_B = \frac{1}{2}(\psi_1 - \psi_2)$$

Describe the physical meaning of variable ψ_B.

c. The linearised equation that governs the evolution of variable ψ_B reads

$$\frac{\partial}{\partial t}\left(\nabla^2\psi_B - \frac{1}{2R^2}\psi_B\right) + \beta_0\frac{\partial\psi_B}{\partial x} = 0\,.$$

Substitute wave-like solutions in this equation for ψ_b and derive the dispersion relation of these waves.
What is the name of these waves?

d. Under certain conditions the two-layer quasi-gesotrophic model describes wave-like solutions of which the amplitude grows exponentially in time.
What conditions are necessary to find such solutions?
Also, give the name of the underlying physical mechanism.

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