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In 2010-2011, the course NS-355b was given by .
The solutions were made by
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Solutions[1] Thermische fysica 2 (NS-355b) November 9, 2010

Question 1: Energy, entropy and free energy

a) If $\mu > 0$, it costs energy to place a particle on the lattice, so the ground states contains no particles:
If $\mu < 0$, it gives energy to place a particle on the lattice, so the ground states contains the maximum number of particles, namely 3:

b) If $\mu = 0$, every states has equal energy, so every state is a ground state, so

$$S = k_B \log \Omega = k_B \log 13 \tag{1}$$

where Ω is the multiplicity.

c) If the temperature is high ($\beta\mu \approx 0$), every state is equally probable, so

$$\langle N \rangle = \frac{1}{13}(1 \times 0 + 5 \times 1 + 6 \times 2 + 1 \times 3) = \frac{20}{13} \tag{2}$$

and

$$\langle N^2 \rangle = \frac{1}{13}(1 \times 0^2 + 5 \times 1^2 + 6 \times 2^2 + 1 \times 3^2) = \frac{38}{13} \tag{3}$$

d) Within linear response theory, $\frac{\partial \langle N \rangle}{\partial \beta\mu} = -\left(\langle N^2 \rangle - \langle N \rangle^2\right)$, so $\frac{\partial \langle N \rangle}{\partial \mu} = \frac{-1}{k_B T}\frac{94}{169}$.

Question 2: polymers

a) In general is $R_{ee}^2 = \langle < |\vec{x}_N - \vec{x}_0|^2 \rangle >= \sum_{i=1}^N \sum_{j=1}^N \langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle$. Since the orientation of connected segments is arbitrary, $\langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle = \delta_{ij} l^2$, so $R_{ee}^2 = \sum_{i=1}^N \sum_{j=1}^N \delta_{ij} l^2 = N l^2$.

b) The energy is minimal if $\vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$ is maximal. The maximum of this inner product is l^2, so $E_{\text{ground state}} = -\epsilon(N-1)$.

c) The probability to find a hinge with angle θ is proportional to it's Boltzman weight;

$$\begin{aligned} P(\theta) \propto e^{-\beta E} &= e^{\beta \frac{\epsilon}{l^2} \vec{\sigma}_i \cdot \vec{\sigma}_{i-1}} && (4)\\ &= e^{\beta \frac{\epsilon}{l^2} l^2 \cos\theta} && (5)\\ &= e^{\beta(E_{\text{ground state}} - \frac{\epsilon}{2}\theta^2)} && (6)\\ &\propto e^{-\frac{\beta\epsilon}{2}\theta^2} && (7) \end{aligned}$$

So θ is normally distributed with standard deviation $\sqrt{\frac{1}{\beta\epsilon}}$, so $A = 1$.

d) Since the angle is normally distributed, you can add the angles quadratically, so $\langle \theta_{i,i+j}^2 \rangle = j \langle \theta_{i,i+1}^2 \rangle$. This means that

$$\begin{aligned} \langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle &= l^2 \langle \cos\theta_{i,i+j} \rangle = l^2 e^{-\frac{j\langle \theta_{i,i+1}^2 \rangle}{2}} && (8)\\ &= l^2 e^{-j\frac{1}{2\beta\epsilon}} && (9) \end{aligned}$$

[1] These solutions were made with great precaution. In case of errors, the $\mathcal{T}_{\mathcal{B}}\mathcal{C}$ cannot be held responsible. However, she will be glad to be informed: `tbc@a-eskwadraat.nl`

Note that in the exam, it was abusively stated that $\langle\cos\theta\rangle \approx \exp(-\langle\theta^2\rangle)$ instead of $\langle\cos\theta\rangle \approx \exp(-\frac{\langle\theta^2\rangle}{2})$. You can still get full points if you used the first relation.

e) The same analysis as in (a) yields to

$$
\begin{aligned}
R_{ee}^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\langle\vec{\sigma}_i\cdot\vec{\sigma}_j\rangle && (10)\\
&= l^2\sum_{i=1}^{N}\sum_{j=1}^{N} e^{-|i-j|\frac{1}{2\beta\epsilon}} && (11)
\end{aligned}
$$

This can be calculated using geometric series:

$$
\begin{aligned}
R_{ee}^2 &= l^2\left(N + 2\frac{1}{1-e^{-B}}\left(Ne^{-B} - \frac{1-e^{-(N+1)B}}{1-e^{-B}}\right)\right) && (12)\\
&\approx l^2\left(N + 2\frac{Ne^{-B}}{1-e^{-B}}\right) = l^2 N\frac{1+e^{-B}}{1-e^{-B}} && (13)
\end{aligned}
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