Instituut voor Theoretische Fysica, Universiteit Utrecht

FINAL EXAM ADVANCED QUANTUM MECHANICS

January 31, 2012

- The duration of the exam is 3 hours.
- The exam is closed-book.
- Usage of a calculator and a dictionary is allowed.
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Problem 1 (20 points)

Consider two particles having only spin degrees of freedom (no orbital angular momentum), of spin s_1 and s_2, that are subject to the following Hamiltonian:

$$H = \alpha + \frac{\beta}{\hbar^2}\vec{S}_1 \cdot \vec{S}_2 + \frac{\gamma}{\hbar}(S_{1,z} + S_{2,z}) \, .$$

Expanding the Hilbert space of the two particle system on a suitable basis, compute the spectrum of the system (energies and eigenstates) in the following three cases:

1. The two particles are distinguishable, and have spin $s_1 = s_2 = 1/2$.
2. The two particles are indistinguishable, and have spin $s_1 = s_2 = 1/2$.
3. The first particle has spin $s_1 = 1/2$ whereas the second has spin $s_2 = 1$.

Problem 2 (20 points)

Consider a particle in a harmonic oscillator potential, whose Hamiltonian can be written in the position representation as follows

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \, .$$

As well known, this system admits an exact solution. However, in this problem you will be asked to use the variational method to find an approximate ground state from the ansatz

$$\psi_a(x) = \frac{1}{x^2 + a} \, , \qquad a > 0 \, .$$

1. Show that the value of the energy functional on the ansatz $\psi_a(x)$ is

$$E[\psi_a] = \frac{\hbar^2}{4m}\frac{1}{a} + \frac{1}{2}m\omega^2 a\,.$$

Note: to solve this point, the following integrals may be useful:

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} = \int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+1)^2} = \frac{\pi}{2}\,,$$

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^4} = \frac{5\pi}{16}\,, \qquad \int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+1)^4} = \frac{\pi}{16}\,.$$

2. Compute, using the variational method, the ground state energy and estimate the error ΔE of the approximation,

$$\Delta E = \frac{E^{\text{variat.}} - E^{\text{exact}}}{\hbar\omega}\,.$$

3. Explain why *any* good ansatz for the ground state of this Hamiltonian should consist of functions of a definite parity (odd or even), i.e. satisfying either

$$f_a^{\text{odd}}(x) = -f_a^{\text{odd}}(-x) \qquad \text{or} \qquad f_a^{\text{even}}(x) = f_a^{\text{even}}(-x)\,.$$

Problem 3 (20 points)

Consider an one-dimensional harmonic oscillator perturbed by a cubic potential. The system is described by the following Hamiltonian,

$$H_\lambda = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2\,Q^2 + \lambda\,V(Q)\,, \qquad V(Q) = \sqrt{\frac{m^3\omega^5}{\hbar}}\,Q^3\,,$$

where $\lambda \ll 1$. As usual, we indicate the eigenstates of the unperturbed problem by $\{|n\rangle\}_{n\in\mathbb{N}}$, meaning that

$$H_0\,|n\rangle = E_n^{(0)}\,|n\rangle \qquad n = 0, 1, 2\ldots\,,$$

in such a way that $E_n^{(0)}$ is an increasing sequence.

1. Show that the matrix element of the potential is

$$\langle m|V(Q)|n\rangle = \left(\frac{(n+3)(n+2)(n+1)}{8}\right)^{1/2} \hbar\omega\, \delta_{m,n+3} + 3\left(\frac{n+1}{2}\right)^{3/2} \hbar\omega\, \delta_{m,n+1}\,,$$

when $m \geq n$.

Note: it can be useful to use the raising and lowering operators, $a^\dagger$ and a; recall that they are related to P and Q by

$$a^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}Q - \frac{i}{\sqrt{m\omega}}P\right), \qquad a = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}Q + \frac{i}{\sqrt{m\omega}}P\right).$$

2. Using perturbation theory, show that the first order correction to the energy vanishes for any state.

3. Using perturbation theory, give an expression for any energy level at orders λ^0, λ^1 and λ^2.

Problem 4 (20 points)

Answer the following points in a clear and concise way.

1. Define the following concepts:
   (a) group,
   (b) representation of a group on a Hilbert space,
   (c) irreducible representation of a group on a Hilbert space,
   (d) unitary representation of a group on a Hilbert space.

2. Consider a group G and a Hilbert space of a quantum mechanical system hosting a unitary representation $g \mapsto U(g)$ of G, where $g \in G$. Prove that, if $U(g)$ commutes with the Hamiltonian for any $g \in G$, then any two states in the same irreducible representation of G have the same energy.

Problem 5 (20 points)

Consider scattering of a one-dimensional particle by a rectangular potential barrier

$$V(x) = \begin{cases} V_0, & |x| \le a \qquad V_0 > 0 \\ 0, & |x| > a\,, \end{cases}$$

assuming that the particle moves towards the potential barrier from the left. Split the real line of x into three regions and choose in regions **I** and **II** a solution $\psi(x,k)$ of the scattering type

- Region **I**: $x < -a, \qquad \psi(x,k) = e^{ikx} + A(k)e^{-ikx}$;
- Region **II**: $x > a, \qquad \psi(x,k) = B(k)e^{ikx}$;
- Region **III**: $-a < x < a$.

1. Solve the Schrödinger equation in the region **III**.
2. By using the sewing conditions for the wave function, show that the scattering coefficients are

   $$A(k) = \frac{e^{-2iak}(e^{4ia\alpha} - 1)(k^2 - \alpha^2)}{e^{4ia\alpha}(k-\alpha)^2 - (k+\alpha)^2}\,, \qquad B(k) = -\frac{4e^{-2ia(k-\alpha)}k\alpha}{e^{4ia\alpha}(k-\alpha)^2 - (k+\alpha)^2}\,,$$

   where $\alpha = \sqrt{k^2 - V_0}$.
3. Determine the value of the transmission and reflection probabilities for the particular case $k^2 = V_0$.