Instituut voor Theoretische Fysica, Universiteit Utrecht

MID-TERM EXAM
ADVANCED QUANTUM MECHANICS

November 8, 2012

- The duration of the exam is 3 hours.
- The exam is closed-book.
- Usage of a calculator and a dictionary is allowed.
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Indicate on the first sheet how many problems you hand in.
- Divide your available time wisely over the exercises.

Problem 1 (10 points)

Consider a state of a particle on a line described by the following wave function

$$\psi(x) = Ce^{\frac{ip_0 x}{\hbar}}\phi(x)\,,$$

where $\phi(x)$ is a real-valued square-integrable function. Show that p_0 coincides with an average momentum of a particle in this state.

Problem 2 (15 points)

Consider a one-dimensional quantum-mechanical problem for a particle of mass m confined in a potential of the form

$$V(x) = \begin{cases} -V_0 + \frac{1}{2}m\omega^2 x^2 & |x| < a, \\ 0 & |x| > a \end{cases}$$

where $V_0 > 0$. We will refer to the region $|x| < a$ as "inside the (harmonic) well". We consider wave functions of the form:

$$\psi(x) = \begin{cases} \psi_\pm(x) = N_\pm \exp\left(\pm\frac{x^2}{2x_0^2}\right) & |x| < a, \\ \text{to be determined} & |x| > a\,. \end{cases}$$

where $N_\pm$ are normalization constants.

1. Determine conditions on x_0 such that $\psi_\pm(x)$ solve the stationary Schrödinger equation inside the well ($|x| < a$). For both ψ_+ and ψ_- write the condition on V_0 such that there are bound state ($E < 0$) solutions, and write the energy eigenvalue E as function of the other parameters (m, ω, V_0).

2. Find the solution of the Schroedinger for a bound state ($E < 0$) on $|x| > a$, where $V(x) \equiv 0$.

3. Match $\psi_-(x)$ properly at $|x| = a$ with the solution you found in the previous point (requiring continuity and differentiability) in order to find the bound state solution on the whole real line. Write this solution up to an overall normalization constant.

   Hint. You will find constraints on the coefficients of the functions inside and outside the well, that have nontrivial solutions only for some values of the energy E.

4. Explain why $\psi_+(x)$ cannot be continued to be an eigenfunction of a bound state on the whole real line.

Problem 3 (20)

Consider an isotropic quantum harmonic oscillator in two dimensions, that is one where the elastic constant is the same in any direction. Furthermore, for simplicity let

$$m = \omega_x = \omega_y = \hbar = 1\,.$$

1. Write the Hamiltonian operator in term of abstract operators P_i, Q_i with $i = x, y$, and explain why this system can be treated as two copies of the one-dimensional problem. Express the Hamiltonian in the position representation, in terms of cartesian coordinates (x, y).

2. Using your knowledge of the one-dimensional case, write what are the allowed energies in the spectrum and give the form of the eigenfunctions.

3. Rewrite the Hamiltonian in the position representation in polar coordinates (r, θ) that satisfy as usual the relations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

Explain why it is possible to separate the angular problem from the radial one and do so.

4. Solve the angular problem by finding eigenfunctions and eigenvalues.

5. Argue whether you expect the eigenvalues of the radial problem to be quantized and/or degenerate (that is, whether some eigenvalues are repeated). In this case, write what their multiplicity is.

   Hint. You do not need to solve the radial problem to answer.

Problem 4 (25 points)

Consider the following Weyl operators

$$U(u) = e^{-iuP}, \qquad V(v) = e^{-ivQ},$$

where u and v are two real numbers (parameters) and (P, Q) are the operators of momentum and coordinate satisfying the Heisenberg commutation relations. The Weyl quantization map associates to a real function $f \equiv f(p, q)$ the following self-adjoint operator A_f

$$A_f = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(u, v)\, U(u)V(v) e^{-\frac{i\hbar uv}{2}}\, du dv\,,$$

where $\hat{f}(u, v)$ is the Fourier image of $f(p, q)$. The inversion formula $A_f \to f$ is

$$f(p, q) = \frac{\hbar}{2\pi} \int_{\mathbb{R}^2} \mathrm{Tr}\left[A_f V(-v)U(-u)\right] e^{\frac{i\hbar uv}{2} - ipu - iqv}\, du dv\,.$$

Let A_f coincide with the projector P_ψ on a state ψ, that is the density matrix $A_f = P_\psi$ is a pure state of quantum mechanics. For any vector φ the projector P_ψ is defined as

$$P_\psi \varphi = (\psi, \varphi)\psi\,.$$

1. Compute the function $\rho_1(p, q)$ which corresponds to the operator P_ψ and further determine the probability distribution density $\rho(p, q) = \frac{\rho_1(p,q)}{2\pi\hbar}$. For simplicity consider ψ in the coordinate representation.

2. Consider the limit $\hbar \to 0$ of $\rho(p, q)$. Does a state of a classical mechanics you obtain in this way is pure or mixed?

Problem 5 (30 points)

Consider the operator A on the domain $D(A) \subset \mathcal{H} = L^2([-1,1])$ defined by

$$\begin{aligned} A\psi\,(x) &= \frac{d}{dx}\left[(x^2-1)\frac{d}{dx}\psi(x)\right], \\ D(A) &= \{\psi \in \mathcal{H} :\ \psi \text{ suitably smooth}\}\,. \end{aligned}$$

Notice that we do not impose boundary conditions on $\psi \in D(A)$.

1. Show that A is self-adjoint (do not consider smoothness issues).

2. The normalized eigenfunctions of this operator are familiar, and given by the formula

$$\psi_n(x) = \sqrt{\frac{2n+1}{2}}\frac{1}{2^n\, n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right], \quad n = 0, 1, 2, \ldots.$$

State the properties of the family of functions $\{\psi_n\}_{n\in\mathbb{N}}$, and say to which quantum mechanical problems they are relevant. Say what their eigenvalues λ_n are.

Hint. If you do not know the eigenvalues by heart are you can derive them by using with operator A on ψ_n (the overall normalization of ψ_n is not important):

$$A\psi_n = \lambda_n\psi_n\,.$$

To find the action of A on ψ_n you may find helpful to use the following formulae

$$\left[x, \frac{d^n}{dx^n}\right] = -n\frac{d^{n-1}}{dx^{n-1}}, \qquad \left[x^2, \frac{d^n}{dx^n}\right] = -2n\frac{d^{n-1}}{dx^{n-1}}x + n(n-1)\frac{d^{n-2}}{dx^{n-2}}\,.$$

3. Let $\ell^2_{\mathbb{N}}$ be the Hilbert space of sequences of complex numbers that are square summable,

$$a \in \ell^2_{\mathbb{N}} \quad \Leftrightarrow \quad a = (a_n)_{n\in\mathbb{N}}, \quad a_n \in \mathbb{C}, \quad \sum_{n=0}^{\infty}|a_n|^2 < \infty,$$

equipped with the scalar product

$$(a,b) = \sum_{n=0}^{\infty} a_n^* b_n\,.$$

Explain how $\{\psi_n\}_{n\in\mathbb{N}}$ mentioned in the previous question can be used to construct a (canonical) isomorphism between $L^2([-1,1])$ and $\ell^2_{\mathbb{N}}$. Find the sequences $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$, with $a, b \in \ell^2_{\mathbb{N}}$ corresponding by this isomorphism to $\psi, \phi \in L^2([-1,1])$ respectively, where $\psi(x) = 1$ and $\phi(x) = x + 2i$.

4. Use Bonnet's formula

$$\frac{n+1}{\sqrt{2n+3}}\psi_{n+1}(x) = \sqrt{2n+1}\, x\, \psi_n(x) - \frac{n}{\sqrt{2n-1}}\, \psi_{n-1}(x), \qquad n = 0, 1, 2, \dots,$$

to find the matrices on $\ell^2_{\mathbb{N}}$ corresponding to the operators B, X, C acting as

$$\begin{aligned} B\psi\,(x) &= 2i\,\psi(x) \\ X\psi\,(x) &= x\,\psi(x) \\ C\psi\,(x) &= (x+2i)\psi(x) \end{aligned}$$

on $L^2([-1,1])$. Explain how the sequences $a, b \in \ell^2_{\mathbb{N}}$ of the previous point are related among themselves in terms of these matrices.