Instituut voor Theoretische Fysica, Universiteit Utrecht

FINAL EXAM ADVANCED QUANTUM MECHANICS

January 29, 2013

- The duration of the exam is 3 hours.
- The exam is closed-book.
- Usage of a calculator and a dictionary is allowed.
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Problem 1 (10 points)

Find the energy levels of the harmonic oscillator

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$$

by using the Bohr-Sommerfeld quantization rule.

Problem 2 (10 points)

The components of the angular momentum are realized as the following operators

$$L_i = \epsilon_{ijk}\, Q_j P_k \,.$$

By using the Heisenberg commutation relations

$$[Q_i, Q_j] = 0\,, \quad [P_i, P_j] = 0\,, \quad [Q_i, P_j] = i\hbar\delta_{ij} I\,,$$

show that for any $i = 1, 2, 3$ the following commutation relations hold

1. $[L_i, P_1^2 + P_2^2 + P_3^2] = 0$;
2. $[L_i, Q_1^2 + Q_2^2 + Q_3^2] = 0$.

Problem 3 (20 points)

Consider a system of two non-interacting electrons, each of them has spin 1/2. Let V be a two-dimensional Hilbert space associated with the spin states of one electron.

1. Determine the dimension of the Hilbert space of this two-electron system;
2. Write down the Lie algebra generators J_i, $i = 1, 2, 3$, of the rotation group acting on the space $V \otimes V$;
3. Compute the corresponding Casimir operator $\vec{J}^2$, where $\vec{J} = (J_1, J_2, J_3)$;
4. What are the spins of the irreducible representations arising in the decomposition of the tensor product $V \otimes V$? Motivate your answer.

Problem 4 (20 points)

Consider a one-dimensional harmonic oscillator perturbed by a quartic potential. The system is described by the following Hamiltonian,

$$H_\lambda = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2 + \lambda\frac{m^2\omega^3}{\hbar}Q^4\,,$$

where $\lambda \ll 1$. As usual, we indicate the eigenstates of the unperturbed problem by $\{|n\rangle\}_{n\in\mathbb{N}}$, meaning that

$$H_0\,|n\rangle = E_n^{(0)}\,|n\rangle \qquad n = 0, 1, 2 \ldots\,,$$

in such a way that $E_n^{(0)}$ is an increasing sequence.

Using perturbation theory, find the first order correction to the energy $E_n^{(0)}$ for an arbitrary n.

Note: it can be useful to use the raising and lowering operators, $a^\dagger$ and a; recall that they are related to P and Q by

$$a^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}Q - \frac{i}{\sqrt{m\omega}}P\right), \qquad a = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}Q + \frac{i}{\sqrt{m\omega}}P\right).$$

Problem 5 (40 points)

In the leading Born approximation

1. Compute the differential cross-section for scattering on the potential $V(r) = V_0\, e^{-\alpha r}$, where V_0 and $\alpha > 0$ are constants and r is a radial coordinate in three dimensions.
2. Compute the corresponding total cross-section.

Problem 6 (Bonus problem) (30 points)

The Hamiltonian of a charged massive particle in an electromagnetic field has the form

$$H[\vec{A},\varphi] = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + \frac{ie\hbar}{mc}(\vec{A}\cdot\vec{\nabla} + \tfrac{1}{2}\mathrm{div}\vec{A}) + \frac{e^2}{2mc^2}\vec{A}^2 + e\varphi\,.$$

Here m and e are the mass and charge of the particle, $\vec{A}$ and φ denote the vector and scalar potential of the electromagnetic field, respectively. Under the gauge transformations the vector and scalar potentials transform as follows

$$\begin{aligned} \vec{A} &\rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi\,, \\ \varphi &\rightarrow \varphi' = \varphi - \tfrac{1}{c}\dot{\chi}\,, \end{aligned}$$

where χ is an arbitrary time-independent function of coordinates and time.

1. Show that if the gauge transformation are supplemented by a simultaneous transformation of the wave function

   $$\psi(x) \rightarrow \psi'(\vec{x}) = e^{i\alpha}\psi(\vec{x})$$

   with the phase $\alpha \equiv \alpha(\vec{x},t)$ being a properly chosen function of coordinates and time, then the Schrödinder equation

   $$i\hbar\frac{\partial\psi'}{\partial t} = H[\vec{A}',\varphi']\psi'$$

   is satisfied as a consequence of the corresponding equation for ψ

   $$i\hbar\frac{\partial\psi}{\partial t} = H[\vec{A},\varphi]\psi\,.$$

2. Show that the density of the probability current

   $$\vec{s} = \frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^* - 2i\frac{e}{\hbar c}\vec{A}\psi^*\psi\right)$$

   is preserved under gauge transformations.