Instituut voor Theoretische Fysica, Universiteit Utrecht

RETAKE EXAM ADVANCED QUANTUM MECHANICS

March 12, 2013

- The duration of the exam is 3 hours.
- The exam is closed-book.
- Usage of a calculator and a dictionary is allowed.
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Part I (Midterm retake)

Problem 1 (25 points)

Consider a particle in a "potential box". The potential energy of the particle is $V = \infty$ for $x < 0$ and $x > a$, and

$$V = 0 \quad \text{for} \quad 0 < x < a\,.$$

For this quantum-mechanical model

1. Find the energy levels E_n and the corresponding normalized wave functions $\psi_n(x)$;
2. Determine the average value $\bar{x}$ of the particle coordinate and its variance

$$\Delta_n^2 x = \langle\psi_n|x^2|\psi_n\rangle - \langle\psi_n|x|\psi_n\rangle^2$$

in the state ψ_n.

Problem 2 (15 points)

Consider the following operators ($-\infty < x < \infty$)

1. Inversion I: $(I\psi)(x) = \psi(-x)$;
2. Shift T_a: $(T_a\psi)(x) = \psi(x+a)$;
3. Dilatation $(D_a\psi)(x) = \sqrt{a}\,\psi(ax), \quad a > 0$;
4. Complex conjugation $(K\psi)(x) = \psi^*(x)$.

Questions:

1. Which of these operators are linear?
2. Find operators which are complex conjugate to the operators above;
3. When it is possible, find operators which are hermitian conjugate to the operators above;
4. Find operators which are inverse to the operators above.

Problem 3 (30 points)

Consider the following Weyl operators

$$\begin{aligned} U(u) &= e^{-iuP}, \\ V(v) &= e^{-ivQ}, \end{aligned}$$

where u and v are two real numbers (parameters) and (P, Q) are the operators of momentum and coordinate satisfying the Heisenberg commutation relations. The Weyl quantization map associates to a real function $f \equiv f(p, q)$ the following self-adjoint operator

$$A_f = \frac{1}{2\pi} \int_{\mathbf{R}^2} \mathrm{d}u \mathrm{d}v \, \hat{f}(u, v) \, e^{\frac{i\hbar uv}{2}} \, V(v) U(u) \, ,$$

where $\hat{f}(u, v)$ is the Fourier image of $f(p, q)$.

1. Find the action of $U(u)$ and $V(v)$ on a wave function in the momentum representation.
2. Find the kernel of the operator A_f in the momentum representation.

Problem 4 (15 points)

Compute the matrix element of the evolution operator of a free particle in one dimension

$$\langle q_2 | e^{-\frac{i}{\hbar}(t_2 - t_1)H} | q_1 \rangle$$

for the transition between a state $|q_1\rangle$ at t_1 and a state $|q_2\rangle$ at t_2, where $|q_1\rangle$ is an eigenstate of the operator of coordinate. The free Hamiltonian is

$$H = \frac{P^2}{2m} \, .$$

Problem 5 (15 points)

Consider the time evolution of a free one-dimensional wave packet

$$\psi(t) = e^{-\frac{i}{\hbar}Ht}\psi\,, \quad H = \frac{P^2}{2m}\,,$$

where the vector $\psi \equiv \psi(0)$ has the following shape in the momentum representation

$$\psi(p,0) = \left(\frac{1}{\pi\alpha^2}\right)^{\frac{1}{4}} e^{-\frac{p^2}{2\alpha^2}}\,,$$

where α is a constant. Find the shape $\psi(x,t)$ of this wave packet at an arbitrary moment of time t in the coordinate representation.