Instituut voor Theoretische Fysica, Universiteit Utrecht

RETAKE EXAM ADVANCED QUANTUM MECHANICS

March 12, 2013

- The duration of the exam is 3 hours.
- The exam is closed-book.
- Usage of a calculator and a dictionary is allowed.
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Retake of Parts I & II (both midterm and final exam)

Problem 1 (30 points)

Consider the following Weyl operators

$$\begin{aligned} U(u) &= e^{-iuP}, \\ V(v) &= e^{-ivQ}, \end{aligned}$$

where u and v are two real numbers (parameters) and (P, Q) are the operators of momentum and coordinate satisfying the Heisenberg commutation relations. The Weyl quantization map associates to a real function $f \equiv f(p,q)$ the following self-adjoint operator

$$A_f = \frac{1}{2\pi} \int_{\mathbf{R}^2} \mathrm{d}u \mathrm{d}v \, \hat{f}(u,v) \, e^{\frac{i\hbar uv}{2}} \, V(v) U(u) \,,$$

where $\hat{f}(u,v)$ is the Fourier image of $f(p,q)$.

1. Find the action of $U(u)$ and $V(v)$ on a wave function in the momentum representation.
2. Find the kernel of the operator A_f in the momentum representation.

Problem 2 (15 points)

Consider the time evolution of a free one-dimensional wave packet

$$\psi(t) = e^{-\frac{i}{\hbar}Ht}\psi \,, \quad H = \frac{P^2}{2m} \,,$$

where the vector $\psi \equiv \psi(0)$ has the following shape in the momentum representation

$$\psi(p,0) = \left(\frac{1}{\pi\alpha^2}\right)^{\frac{1}{4}} e^{-\frac{p^2}{2\alpha^2}} \,,$$

where α is a constant. Find the shape $\psi(x,t)$ of this wave packet at an arbitrary moment of time t in the coordinate representation.

Problem 3 (10 points)

Consider a state ψ_{lm} with definite values l and m of the angular momentum and its projection on z-axis, respectively. Find the mean values $\overline{L_x^2}$ and $\overline{L_y^2}$ in this state.

Problem 4 (20 points)

Consider a system of two non-interacting particles, one of them has internal spin 1/2 and the other 1. Let $V_{1/2}$ and V_1 be the corresponding Hilbert spaces. For the particle of spin 1/2 the Lie algebra generators of the rotation group acting in the space $V_{1/2}$ are given by Pauli matrices, while for the particle of spin 1 they are realized as the following matrices

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

acting in the space V_1.

1. Determine the dimension of the Hilbert space of this two-particle system;
2. Write down the Lie algebra generators J_i, $i = 1, 2, 3$, of the rotation group acting in the space $V_{1/2} \otimes V_1$;
3. Compute the corresponding Casimir operator $\vec{J}^2$, where $\vec{J} = (J_1, J_2, J_3)$;
4. What are the spins of the irreducible representations arising in the decomposition of the tensor product $V_{1/2} \otimes V_1$? Motivate your answer.

Problem 5 (25 points)

Consider a charged one-dimensional harmonic oscillator in a homogeneous electric field directed along the axis of oscillations. It is described by the following Hamiltonian

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2 - e\mathscr{E}Q,$$

where e is a charge and $\mathscr{E}$ is an electric field.

1. Treating the action of the electric field on the charge as a perturbation, compute in the first two orders of perturbation theory the shift of the energy levels caused by the electric field.
2. By solving the stationary Schrödinger equation for the Hamiltonian H find the exact energy levels. Compare the result obtained by perturbation theory with the exact answer.

Note: it can be useful to use the raising and lowering operators, $a^\dagger$ and a; recall that they are related to P and Q by $a^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}Q - \frac{i}{\sqrt{m\omega}}P\right)$, $a = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}Q + \frac{i}{\sqrt{m\omega}}P\right)$.