Institute for Theoretical Physics
Utrecht University

Final Exam "Advanced Quantum Mechanics" (total of 80 points)

Tuesday, 28 January 2014, 13:30-16:30

1. **USE A SEPARATE SHEET FOR EVERY EXERCISE**
2. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *Give the motivation, explanation, and calculations leading up to each answer and/or solution.*
5. *Do not spend a large amount of time on finding (small) calculational errors. If you suspect you have made such an error, point it out in words.*

Exercise 1: Some concepts (total: 15 points)

a) *(5 points)* At time $t = 0$, a single particle with mass m that moves in one dimension and feels no potential (and is subject to periodic boundary conditions over a length L) is in a state that corresponds to the wave function

$$\psi(x) = \sum_n a_n e^{\frac{2\pi i n x}{L}} ,$$

where the sum is over all integers. At time t a single measurement is performed. Give the normalized wave function corresponding to the state directly after the measurement if i) the magnitude and direction of the momentum are measured with the result p, ii) the energy is measured with the result $p^2/2m$ and iii) the magnitude (but not the direction) of the momentum is measured with the result p.

b) *(3 points)* Give an example of a physical observable that is conserved for i) a translation-invariant system, ii) a rotation-invariant system, and iii) a system with a time-independent hamiltonian.

c) *(2 points)* A spinless ($S = 0$) molecule is described by a 103-dimensional irreducible representation of the rotation group; what does a measurement of the square of the angular momentum yield?

d) *(3 points)* Show that for the wave function describing a system of two identical particles it has to hold that $\Psi(x_1, x_2) = e^{i\theta}\Psi(x_2, x_1)$ where x_1, x_2 are positions of the two particles and θ is real.

e) *(2 points)* Suppose that a system of two particles is, at a certain time, described by the state $|\Psi\rangle$. Give the probability amplitude for finding one particle at position x_1 and another at position x_2 for i) identical bosons, ii) identical fermions, and iii) distinguishable particles.

Exercise 2: Two particles (total: 20 points)

Consider two particles that do not interact with each other in an harmonic-oscilator potential. The single-particle hamiltonian has eigenstates $|n\rangle$ with $n = 0, 1, 2, 3, \cdots$ and eigenenergies $\epsilon_n = (n + 1/2)\hbar\omega$, where ω is the frequency of the harmonic oscillator. Give the normalized physical eigenstates and energies of the two-particle system for i) identical spin $S = 0$ particles, ii) identical spin $S = 1/2$ particles, iii) one spin $S = 1/2$ and one $S = 1$ particle, iv) identical spin $S = 1$ particles. NB: in this exercise assume that there is no magnetic field affecting the spin via a Zeeman interaction (note that this latter statement does not imply anything about the presence/absence of magnetic fields in other exercises).

Exercise 3: Some manipulations (total: 20 points)

a) *(5 points)* Consider the operator $\hat{O} = \hat{A}^\dagger \hat{A}$ where $\hat{A}$ is an arbitrary operator in Hilbert space. Show that the eigenvalues of $\hat{O}$ are positive.

b) *(5 points)* Consider an operator $\hat{O}$ which, in a certain complete basis with elements $|e_j\rangle$ labeled by integer numbers j, has the matrix elements $O_{jk} = \langle e_j|\hat{O}|e_k\rangle$. Consider a different basis $|f_j\rangle = \frac{1}{\sqrt{2}}[|e_j\rangle + |e_{j+1}\rangle]$. Give the matrix elements of $\hat{O}$ in this new basis in terms of O_{jk}.

c) *(10 points)* Consider a single spin S, described by the hamiltonian $\hat{H}$. Introduce the spin raising and lowering operators $\hat{S}_+ = \hat{S}_x + i\hat{S}_y$ and $\hat{S}_- = \hat{S}_x - i\hat{S}_y$. i) Show that the operators

$$\begin{aligned}\hat{S}_+ &= \hbar\sqrt{2S}\hat{a};\\ \hat{S}_- &= \hbar\sqrt{2S}\hat{a}^\dagger;\\ \hat{S}_z &= \hbar(S - \hat{a}^\dagger\hat{a});\end{aligned}$$

with commutator $[\hat{a}, \hat{a}^\dagger] = 1$ obey the angular momentum commutation relations $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$ within the subspace of Hilbert space for which $S \gg \langle\hat{a}^\dagger\hat{a}\rangle$. ii) Give a physical interpretation of this result. Note that the commutation relations of $\hat{a}$ and $\hat{a}^\dagger$ imply that they correspond to lowering and raising operators for a one-dimensional harmonic oscillator.

Exercise 4: Schrödinger equation in the rotating frame (total: 25 points)

Consider a two-dimensional system described by coordinates (x, y), consisting of one spinless particle ($S = 0$) with mass m in a potential $V(x, y)$ that depends only on $r = \sqrt{x^2 + y^2}$. The time-dependent Schrödinger equation describing the wave function of this system is therefore

$$i\hbar\frac{\partial\Psi(x,y,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y)\right]\Psi(x,y,t)\ .$$

Consider now a transformation given by $x \to \cos(\omega t)x + \sin(\omega t)y$ and $y \to \cos(\omega t)y - \sin(\omega t)x$, which therefore corresponds to a transformation to a coordinate frame rotating, with angular frequency ω, with respect to the original frame.

a) *(5 points)* Show that the Schrödinger equation in this rotating frame is

$$i\hbar\frac{\partial\Psi(x,y,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \omega\hat{L}_z + V(x,y)\right]\Psi(x,y,t)\ ,$$

with $\hat{L}_z = -i\hbar(x\partial/\partial y - y\partial/\partial x)$ the operator corresponding to the angular momentum in the z-direction.

b) *(10 points)* Show that the Schrödinger equation can be written in the form of that corresponding to a particle with charge q in an electromagnetic field so that

$$i\hbar\frac{\partial\Psi(x,y,t)}{\partial t} = \left[\frac{(\hat{\mathbf{p}} - q\mathbf{A})^2}{2m} + V(x,y) + V_\omega(x,y)\right]\Psi(x,y,t)\ ,$$

with $\hat{\mathbf{p}} = -i\hbar(\partial/\partial x, \partial/\partial y, 0)$ the momentum operator, $\mathbf{A} = m\omega\mathbf{e}_z \times (x, y, 0)/q$ an effective vector potential with $\mathbf{e}_z$ the unit vector in the z-direction, and $V_\omega(x, y) = -m\omega^2(x^2 + y^2)/2$.

c) *(10 points)* The above shows that the dynamics of a particle in a rotating frame is effectively the same as that of a charged particle in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. i) Give a physical reason why this is so, and give also the expression for $\mathbf{B}$. ii) Give also a physical interpretation of the potential $V_\omega(x, y)$.

Exercise one

i)

$$\psi(x) = \frac{1}{|a_n|^2\sqrt{L}}\, a_n\, e^{i2\pi n x/L}$$

where n is such that $p = \frac{\hbar\, 2\pi n}{L}$

ii)

$$\psi(x) = \frac{1}{\sqrt{L}}\left(\frac{a_n e^{i2\pi n x/L} + a_{-n} e^{-i2\pi n x/L}}{|a_n|^2 + |a_{-n}|^2}\right)$$

where n is such that $\frac{p^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi n}{L}\right)^2$

iii) See ii) where n is such that $p = \frac{\hbar\, 2\pi\, |n|}{L}$

b)
i) momentum
ii) angular momentum
iii) energy

c) $2L + 1 = 103 \rightarrow L = 51 \rightarrow \langle \vec{L}^2 \rangle = 51(51+1)\hbar^2$
$= 51 \times 53\, \hbar^2$

d) For identical particles $P(x_1, x_2) = P(x_2, x_1)$

where $P(x_1, x_2) = |\psi(x_1, x_2)|^2$

Satisfied if $\psi(x_1, x_2) = e^{i\theta}\, \psi(x_2, x_1)$

xercise 1

e) i)/ii) $\frac{1}{\sqrt{2}}\left[{}_1\langle x_1|\,{}_2\langle x_2| \pm {}_1\langle x_2|\,{}_2\langle x_1|\right]|\psi\rangle$

iii) ${}_1\langle x_1|\,{}_2\langle x_2|\,|\psi\rangle$

Exercise 2

i) $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|n_1\rangle_1 \, |n_2\rangle_2 + |n_2\rangle_1 \, |n_1\rangle_2\right]$

energy $E = \varepsilon_{n_1} + \varepsilon_{n_2}$

when $n_1 = n_2$ ~~fac~~ $|\psi\rangle = |n_1\rangle_1 |n_1\rangle_2$

ii) $|\psi\rangle = |\psi\rangle_{orb} \otimes |\psi_{spin}\rangle$

where $|\psi_{orb}\rangle = \frac{1}{\sqrt{2}}\left[|n_1\rangle_1 |n_2\rangle_1 - |n_2\rangle_1 |n_1\rangle_2\right]$

should be multiplied with $|\psi_{spin}\rangle = \begin{cases} |\uparrow\rangle_1 |\uparrow\rangle_2 \\ |\downarrow\rangle_1 |\downarrow\rangle_1 \\ \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle\right. \end{cases}$

or $|\psi_{orb}\rangle = \frac{1}{\sqrt{2}}\left[|n_1\rangle_1 |n_2\rangle_2 + |n_2\rangle_1 |n_1\rangle_2\right]$ (when $n_1 = n_2$, see i) for norm

is combined with $|\psi_{spin}\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle\right.$

iii) $|\psi\rangle = |\sigma\rangle_1 |n_1\rangle_1 \; |1; m_S\rangle_2 \, |n_2\rangle_2$

where ~~t~~ $\sigma \in \{\uparrow, \downarrow\}$ and $m_S \in \{-1, 0, 1\}$

iv) $|\psi\rangle = |\psi_{orb}\rangle \otimes |\psi_{spin}\rangle$

S (normalization if $n_1 = n_2$, see i

where $|\psi_{orb}^{S/A}\rangle = \frac{1}{\sqrt{2}}\left[|n_1\rangle_1 |n_2\rangle_2 \pm |n_2\rangle_1 |n_1\rangle_1\right]$

A

The ~~xx~~ symmetric (S) and antisymmetric orbital state should be combined with and ~~anti~~symmetric and antisymmetric spin part, respectively; ~~that~~

$$|\psi^{S/A}_{spin}\rangle = \frac{1}{\sqrt{2}} |1; m_{S_1}\rangle_1 \, |1; m_{S_2}\rangle \underset{A}{\overset{S}{\pm}} |1; m_{S_2}\rangle_1 |1; m$$

(where normalization for S state changes if $m_{S_1} = m_{S_2}$)

3a)

Assume $A^\dagger A|\psi\rangle = \lambda|\psi\rangle$

then

$$\langle\psi|A^\dagger A|\psi\rangle = \lambda\langle\psi|\psi\rangle$$

$$\rightarrow \langle A\psi|A\psi\rangle = \lambda \quad \text{if } |\psi\rangle \text{ normalized}$$

$$\geq 0 \quad \text{so} \quad \lambda \geq 0$$

Exercise

a) $\hat{O}^\dagger = \hat{O}$ is herm.

$\langle d_j|\hat{O}|d_k\rangle = \frac{1}{2}\big(\langle e_j| + \langle e_{j+1}|\big)\big(\hat{O}\,\big(|e_k\rangle + |e_{k+1}\rangle\big)\big)$

$= \frac{1}{2}\big[\langle e_j|\hat{O}|e_k\rangle + \langle e_j|\hat{O}|e_{k+1}\rangle + \langle e_{j+1}|\hat{O}|e_k\rangle + \langle e_{j+1}|\hat{O}|e_{k+1}\rangle\big]$

$= \frac{1}{2}\big[O_{jk} + O_{j,k+1} + O_{j+1,k} + O_{j+1,k+1}\big]$

b)

$S_x = \frac{S_+ + S_-}{2}$, $S_y = \ldots$

c) i) $[S_x, S_y] = \frac{1}{4i}\big[S_+ + S_-, \ldots\big]$

So $[S_x, S_y] = \frac{1}{4i}\big([S_+, -S_-] + [S_-, S_+]\big)$

$= \frac{1}{2i}[S_-, S_+]$

$= \frac{1}{2i}\hbar^2\, 2S\, [a^\dagger \ldots$

$= \frac{\hbar^2\, 2S}{2i} \ldots$

b) $\langle f_j | \hat{O} | f_k \rangle = \frac{1}{2}\Big(\langle e_j | + \langle e_{j+1} | \Big) | \hat{O} |$

$\Big(| e_k \rangle + | e_{k+1} \rangle \Big)$

$= \frac{1}{2}\Big[\langle e_j | \hat{O} | e_k \rangle + \langle e_j | \hat{O} | e_{k+1} \rangle + \langle e_{j+1} | \hat{O} | e_k \rangle$

$+ \langle e_{j+1} | \hat{O} | e_{k+1} \rangle \Big]$

$= \frac{1}{2} \Big[O_{jk} + O_{jk+1} + O_{j+1,k} + O_{j+1\,k+1} \Big]$

c) i) $S_x = \frac{(S_+ + S_-)}{2}$; $S_y = \frac{S_+ - S_-}{2i}$

So $[S_x, S_y] \equiv \frac{1}{4i} [S_+ + S_-, S_+ - S_-]$

$= \frac{1}{4i} \Big([S_+, -S_-] + [S_-, S_+] \Big)$

$= \frac{1}{2i} [S_-, S_+]$ and $[S_-, S_+]$

$= \frac{\hbar^2 2S}{2i} [a^\dagger, a] = + i\hbar^2 S$ since $[a, a^\dagger] = 1$

Thus:

$$[S_x, S_y] = i\hbar^2 S = i\hbar S_z$$

provide S_z is evaluated in a part of Hilbert space

where: $\hat{S}_z = \hbar(S - a^\dagger a) \simeq \hbar S$ so that

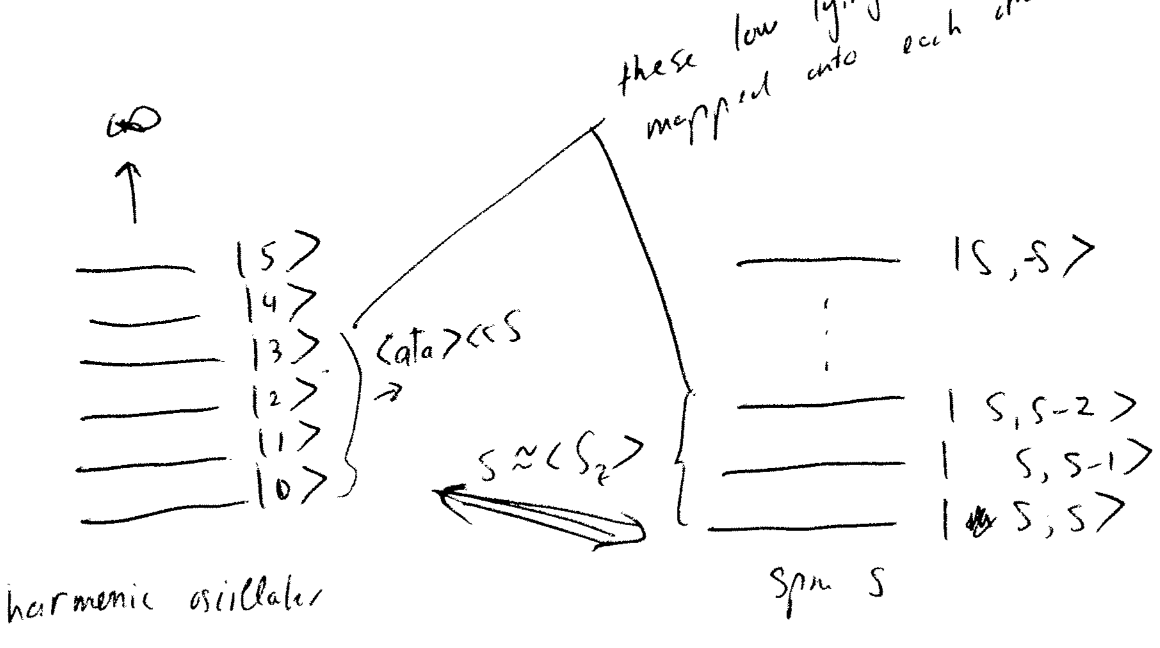
$\langle a^\dagger a \rangle \ll S$ for that part of Hilbert space.

(i) If $\langle a^\dagger a \rangle \ll S$ then only a few eigenstates of ~~the~~ S_z are important, ie, $|S; m_s\rangle$ with $m_s \gg -S$

Similarly, if $\langle a^\dagger a \rangle$ is small only a few h.o. states are important.

The operator mapping essentially maps these states

In pictures:



Exercise 4 — -1 per important step missing

a) Method 1:

$$i\hbar \frac{\partial}{\partial t}\psi(x',y',t) = i\hbar \frac{d}{dt}\psi(x\cos\omega t + y\sin\omega t,\ y\cos\omega t - x\sin\omega t,\ t)$$ (+3)

$$= i\hbar \frac{\partial \psi}{\partial x'}\big(\underbrace{-\omega x\sin\omega t + y\omega\cos\omega t}_{\omega y'}\big)$$

$$+ i\hbar \frac{\partial \psi}{\partial y'}\big(\underbrace{-\omega y\sin\omega t - x\omega\cos\omega t}_{-\omega x'}\big) + i\hbar\frac{\partial \psi}{\partial t}$$

$$= \omega\left[i\hbar\, y'\frac{\partial \psi}{\partial x'} - x'\, i\hbar\frac{\partial \psi}{\partial y'}\right] + i\hbar\frac{\partial \psi}{\partial t}$$

$$= \omega L_z\psi + i\hbar\frac{\partial \psi}{\partial t}$$

The other term remain unaffected, because they depend only on $r = \sqrt{x^2+y^2}$ (+2)

→ after renaming $x' \to x$, $y' \to y$ we have:

$$i\hbar\frac{\partial \psi}{\partial t} + \omega L_z\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi$$

so $$i\hbar\frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 - \omega L_z + V\right]\psi.$$

method 2

We have: $|\psi\rangle \to e^{-\frac{i}{\hbar}(\omega t)\hat{L}_z}|\psi\rangle$ because

of rotation.

So $i\hbar\frac{d}{dt}|\psi\rangle \to i\hbar\frac{d}{dt}\left(e^{-i/\hbar(\omega t)\hat{L}_z}|\psi\rangle\right)$

$$= \cancel{i\hbar}\, e^{-i/\hbar\,\omega t\,\hat{L}_z}\left(i\hbar\frac{d}{dt} + \omega L_z\right)|\psi\rangle$$

So $i\hbar\frac{d}{dt} \to \left(i\hbar\frac{d}{dt} + \omega L_z\right)|\psi\rangle$ +3

b) $$\vec{A} = \frac{m\omega}{q}\hat{z}\times\begin{pmatrix}x\\y\\0\end{pmatrix} = \frac{m\omega}{q}\begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}x\\y\\0\end{pmatrix} = \frac{m\omega}{q}\begin{pmatrix}-y\\x\\0\end{pmatrix}$$ +2

So: $$(p^2 - qA)^2 = \vec{p}^2 + -q\,\vec{p}\cdot\vec{A} + -q\vec{A}\cdot\vec{p} + q^2A^2$$

$$= \vec{p}^2 - 2q\,\vec{A}\cdot\vec{p} + q^2A^2$$

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because this particular A_a and p_α commute.

$$\vec{A}\cdot\vec{p} = \frac{m\omega}{q}\begin{pmatrix}-y\\x\\0\end{pmatrix}\cdot\begin{pmatrix}-i\hbar\frac{\partial}{\partial x}\\-i\hbar\frac{\partial}{\partial y}\\0\end{pmatrix} = \frac{m\omega}{q}(+)i\hbar\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

$$= -\frac{m\omega}{q}L_z$$

2

And $q^2 A^2 = q^2 \frac{m^2\omega^2}{q^2}(y^2+x^2)$ 1

So $$\frac{(p-qA)^2}{2m} = \frac{p^2}{2m} - \omega L_z + \frac{m\omega^2}{2}(x^2+y^2)$$

The last term cancels against $V_\omega(x,y)$ so we get back to the result of part 4 a)

c) i) $\vec{B}$ leads to a Lorentz force +3, which has the same form as the Coriolis force +3 in the rotating frame

$$\vec{B} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 0 \end{pmatrix} \times \frac{m\omega}{q}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{2m\omega}{q} \end{pmatrix} = \frac{2m\omega}{q}\hat{z}$$ +1

ii) Centrifugal force.

+3