INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Final Exam Advanced Quantum Mechanics (total 300 points)

Tuesday, January 31, 2017, 13:30-16:30

1. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
2. *Write clearly, unreadable work cannot be corrected.*
3. *Give the motivation, explanation, and calculations leading up to each answer and/or solution.*
4. *Do not spend a large amount of time on finding (small) calculational errors. If you suspect you have made such an error, point it out in words.*
5. *Note the appendix at the end of this exercise!*
6. *This exam consists of three exercises. Start each exercise on a new sheet of paper.*

1. SOME CONCEPTS (*80 POINTS*)

a) (*30 points*) Describe in a few lines — using a few equations if you want — what is meant by i) entanglement, ii) a mixed state, iii) a pure state, and iv) decoherence.

b) (*10 points*) Describe what a superposition is and/or give an example of a superposition.

c) (*20 points*) Describe briefly what a measurement of violation of the Bell inequalities implies (NB: do not describe how such measurements are done, but, essentially, why they are done).

d) (*10 points*) Give an example of a practical application of entanglement that is currently envisioned, or is already in use.

e) (*10 points*) Give the time-dependent Schrödinger equation for a system described by a time-dependent hamiltonian $\hat{H}(t)$; give an example of a situation where a time-dependent hamiltonian may arise

2. TWO PARTICLES WITH SPIN (*120 POINTS*)

In this exercise, we consider a particle with spin $S = 3/2$. We discard the orbital motion of the particle. Assume that the particle is in the state

$$|\Psi\rangle = \sum_{m_S=-S}^{m_S=S} a_{m_S}|S, m_S\rangle \ , \qquad (1)$$

where the states $|S, m_S\rangle$ are simultaneous eigenstates of $\hat{S}_z$ and $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$, and $a_{-3/2}, a_{-1/2}, a_{1/2}, a_{3/2}$ are complex numbers.

a) (*10 points*) What are the possible outcomes for a measurement of the spin along the x-direction?

b) (*30 points*) Give the expectation value of $\hat{S}_x$ for the state in Eq. (1). (NB: Note the appendix at the end of the exam!)

Assume now that the system is described by the density matrix

$$\hat{\rho} = \sum_{m_S=-S}^{m_S=S} p_{m_S} |S, m_S\rangle\langle S, m_S| \ , \tag{2}$$

with p_{m_S} the probability to be in the state $|S, m_S\rangle$.

c) (*20 points*) Give the expectation value of $\hat{S}_y$ and $\hat{S}_z$ for a system described by the density matrix in Eq. (2)

d) (*30 points*) Consider now the case that the system is in thermal equilibrium with temperature T, and described by the hamiltonian $\hat{H} = K_1\hat{S}_z + K_2\hat{S}_z^2$, with K_1, K_2 constants. Give the expectation value of $\hat{S}_z$.

We consider for the remainder of this exercise two spin-3/2 particles.

e) (*10 points*) Give the possible outcomes for a measurement of the x, y, and z-component of the spin of the system consisting of two particles.

f) (*20 points*) Assume the hamiltonian for the two particles is given by

$$\hat{H} = -\alpha \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \ ,$$

with $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ the operators describing the spin of particle 1 and 2, respectively, and $\alpha > 0$ a constant. Give the eigenvalues of this hamiltonian.

3. SPIN-ORBIT COUPLING (*100 POINTS*)

Consider a three-dimensional electron gas (with periodic boundary conditions) with spin-orbit interactions. The hamiltonian is given by

$$H = \frac{\hat{\mathbf{p}}^2}{2m} - \alpha\hat{\mathbf{p}} \cdot \boldsymbol{\tau} - B\tau_z \ ,$$

where B is a magnetic field, $\alpha > 0$ is a constant, and the Pauli matrices are given by

$$\tau^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \tau^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \tau^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3}$$

Moreover, $\hat{\mathbf{p}}$ is the momentum operator that, in the position representation, is given by $-i\hbar\boldsymbol{\nabla}$, as usual. (Note that in the parts a), b) and c) of this exercise we ignore the influence of the magnetic field on the orbital motion of the electrons.)

a) (*50 points*) Calculate the energy eigenvalues of the hamiltonian, and the corresponding energy eigenstates

$$\chi(\mathbf{x}) = \begin{pmatrix} \chi_\uparrow(\mathbf{x}) \\ \chi_\downarrow(\mathbf{x}) \end{pmatrix} ,$$

in position representation and up to a normalization factor (NB: this means you do not have to normalize the states!).

b) (*20 points*) Determine the velocity operator $\frac{d\hat{\mathbf{x}}}{dt} \equiv \frac{1}{i\hbar}[\hat{\mathbf{x}}, \hat{H}]$, with $[\cdot,\cdot]$ the commutator, and $\hat{\mathbf{x}}$ the position operator.