INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Final Exam Advanced Quantum Mechanics (total 300 points)

Tuesday, January 30, 2018, 13:30-16:30

1. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
2. *Write clearly, unreadable work cannot be corrected.*
3. *Give the motivation, explanation, and calculations leading up to each answer and/or solution.*
4. *Do not spend a large amount of time on finding (small) calculational errors. If you suspect you have made such an error, point it out in words.*
5. *Note the appendix at the end of this exercise!*
6. *This exam consists of three exercises. Start each exercise on a new sheet of paper.*

1. SOME CONCEPTS (*80 POINTS*)

a) (*30 points*) Describe in a few lines — using a few equations if you want — what is meant by i) entanglement, ii) a mixed state, iii) a pure state, and iv) decoherence.

b) (*10 points*) Describe what a superposition is and/or give an example of a superposition.

c) (*20 points*) Describe briefly what a measurement of violation of the Bell inequalities implies (NB: do not describe how such measurements are done, but, essentially, why they are done).

d) (*10 points*) Give an example of a practical application of entanglement that is currently envisioned, or is already in use.

e) (*10 points*) Consider a spin with total spin angular momentum quantum number $S_1 = 4$, and another spin with total spin angular momentum quantum number $S_2 = 3/2$. Give the possible outcomes of a measurement of the total spin angular momentum of a quantum system that consists of both these spins together.

2. PARTICLE WITH SPIN IN A ROTATING FRAME (*100 POINTS*)

Consider a two-dimensional system described by coordinates (x, y), consisting of one spin-1/2 particle ($S = 1/2$) with mass m in a potential $V(x, y)$ that depends only on $r = \sqrt{x^2 + y^2}$. In the basis-independent formulation, the time-dependent Schrödinger equation for the state vector $|\Psi\rangle$ is therefore

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left[\frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + V(\hat{x}, \hat{y})\right]|\Psi\rangle \ .$$

Consider now a transformation to a coordinate frame rotating, with angular frequency ω around the z-axis, with respect to the original frame.

a) *(20 points)* Show that the Schrödinger equation in this rotating frame is

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left[\frac{\hat{p}_x^2+\hat{p}_y^2}{2m} - \omega\left(\hat{L}_z+\hat{S}_z\right) + V(\hat{x},\hat{y})\right]|\Psi\rangle\,,$$

with $\hat{L}_z$ the operator corresponding to the orbital angular momentum in the z-direction, and $\hat{S}_z$ the z-component of the spin angular momentum.

b) *(30 points)* Define the wave functions $\Psi_\sigma(x,y) \equiv (\langle x,y| \otimes \langle\sigma|)|\Psi\rangle$, with $|x,y\rangle$ the eigenstates of the position operator and $|\sigma\rangle$ the eigenstates of $\hat{S}_z$ with respective eigenvalues $\sigma\hbar/2$, and $\sigma \in \{+1,-1\}$. Show that the Schrödinger equation in position representation is given by

$$i\hbar\frac{\partial\Psi_\sigma(x,y,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - \frac{\sigma\hbar\omega}{2} + i\hbar\omega\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + V(x,y)\right]\Psi_\sigma(x,y,t)\,.$$

c) *(30 points)* Show that this latter Schrödinger equation can be written in the form of that corresponding to a particle with charge q in an electromagnetic field so that

$$i\hbar\frac{\partial\Psi_\sigma(x,y,t)}{\partial t} = \left[\frac{(\hat{\mathbf{p}}_{\text{pos}} - q\mathbf{A})^2}{2m} + V(x,y) - \frac{\sigma\hbar\omega}{2} + V_\omega(x,y)\right]\Psi_\sigma(x,y,t)\,,$$

with $\hat{\mathbf{p}}_{\text{pos}} = -i\hbar(\partial/\partial x, \partial/\partial y, 0)$ the momentum operator in the position representation, $\mathbf{A} = m\omega\mathbf{e}_z \times (x,y,0)/q$ an effective vector potential with $\mathbf{e}_z$ the unit vector in the z-direction, and $V_\omega(x,y) = -m\omega^2(x^2+y^2)/2$.

d) *(20 points)* The above shows that the dynamics of a particle in a rotating frame is effectively the same as that of a charged particle in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. i) Give a physical reason why this is so, and give the expression for $\mathbf{B}$. ii) Give a physical interpretation of the potential $V_\omega(x,y)$.

3. ATOMIC NUCLEUS IN AN ELECTRIC FIELD (*120 POINTS*)

Consider a particle with spin s. We ignore its orbital motion and assume its normalized state vector to be

$$|\Psi\rangle = \sum_{m_s=-s}^{s} a_{m_s}|s,m_s\rangle$$

Here, the states $|s,m_s\rangle$ are the simultaneous eigenstates of $\hat{S}_z$ and $\hat{\mathbf{S}}^2$ and the coefficients a_{m_s} are complex numbers.

a) (*15 points*) For the case $s=1$, what are the possible outcomes for a measurement along the y-direction and what are the corresponding probabilities? (NB: Note the appendix at the end of the exam!)

Assume now that an ensemble of such spin s particles is described by the density operator

$$\hat{\rho} = \sum_{m_s=-s}^{s} p_{m_s}|s,m_s\rangle\langle s,m_s|$$

b) (*15 points*) Give the physical meaning of the numbers p_{m_s} and at least one condition which these numbers must satisfy for the case of general s. Give the density operator for the case $s=1$ for i) a pure state, and ii) a uniformly distributed ensemble, and describe their physical meanings.

Consider a Hamiltonian of the form [1]

$$\hat{H} = k \sum_{i,j=1}^{3} \hat{S}_i V_{ij} \hat{S}_j$$

where k is a constant, $\hat{S}_i$ is the spin operator corresponding to the i-th Cartesian direction and V_{ij} are the elements of a real traceless symmetric 3×3 matrix V — note that this latter matrix is not a quantummechanical operator.

c) (*15 points*) Why can you always rotate to new coordinates x, y, z in which V is diagonal?

Hence, in those new coordinates,

$$V = \begin{pmatrix} V_{xx} & 0 & 0 \\ 0 & V_{yy} & 0 \\ 0 & 0 & V_{zz} \end{pmatrix}$$

with $V_{xx} + V_{yy} + V_{zz} = 0$.

d) (*20 points*) Show that the Hamilton operator in these coordinates can be written as

$$\hat{H} = K\left(3\hat{S}_z^2 - \hat{\mathbf{S}}^2 + \eta\,(\hat{S}_x^2 - \hat{S}_y^2)\right) \quad , \quad \eta = \frac{V_{xx} - V_{yy}}{V_{zz}} ,$$

and give K. Here, we assumed that $V_{zz} \neq 0$.

e) (*10 points*) Give the exact eigenvalues and eigenvectors of $\hat{H}$ for the case $\eta = 0$ and any spin s. How many energy levels are there and what is the degree of degeneracy of each level?

f) (*30 points*) Find the exact eigenvalues and normalized eigenvectors of $\hat{H}$ for general η and spin 1 and draw a graph of the energy levels as function of η for $0 \leq \eta \leq 1$. Hint: Start by expressing $\hat{S}_x^2 - \hat{S}_y^2$ in terms of $\hat{S}_\pm = \hat{S}_x \pm \mathrm{i}\hat{S}_y$.

g) (*15 points*) Consider such a spin 1 in thermal equilibrium at temperature T. Find the expectation value of $\hat{S}_z$ for $\eta = 0$.

Appendix — Possibly, you may want to make use of these results:

- Under a rotation of angle θ around an axis along the unit vector $\mathbf{n}$, the state vector transforms according to $|\Psi\rangle \to e^{-\frac{i}{\hbar}\theta \mathbf{n}\cdot\hat{\mathbf{J}}}|\Psi\rangle$, where $\hat{\mathbf{J}}$ is the operator corresponding to total angular momentum.
- For an angular momentum operator $\hat{\mathbf{J}}$ we have the states $|j, m\rangle$, which are the eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$. The respective eigenvalues are $j(j+1)\hbar^2$ and $m\hbar$, where m runs from $-j$ to $+j$ in integer steps and j is an integer. Furthermore, we have that $\hat{J}_\pm = \hat{J}_x \pm \mathrm{i}J_y$ which act on these states as:

$$\hat{J}_+ |j, m\rangle = \hbar\sqrt{j\,(j+1) - m\,(m+1)}\,|j, m+1\rangle\,;$$

$$\hat{J}_- |j, m\rangle = \hbar\sqrt{j\,(j+1) - m\,(m-1)}\,|j, m-1\rangle\,.$$

- For $s = 1$ and in the basis $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$, the spin operators $\hat{S}_i$ are represented by

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad , \quad \hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

---

[1] For example, an atomic nucleus with a non-zero electric quadrupole moment in an inhomogeneous electric field is described by this hamiltonian. In that case V results from the gradient of the electric field.