INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Retake Advanced Quantum Mechanics (total 300 points)

Tuesday, April 17, 2018, 13:30-16:30

1. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
2. *Write clearly, unreadable work cannot be corrected.*
3. *Give the motivation, explanation, and calculations leading up to each answer and/or solution.*
4. *Do not spend a large amount of time on finding (small) calculational errors. If you suspect you have made such an error, point it out in words.*
5. *Note the appendix at the end of this exercise!*
6. *This exam consists of three exercises. Start each exercise on a new sheet of paper.*

1. SOME CONCEPTS (*80 POINTS*)

a) (*30 points*) Describe in a few lines — using a few equations if you want — what is meant by i) a superposition, ii) decoherence, iii) a pure state, and iv) entanglement.

b) (*10 points*) Describe what a mixed state is and/or give an example of a mixed state.

c) (*10 points*) A system is in the state $|\Psi\rangle$ when a physical quantity is measured. This quantity is described by the operator $\hat{O}$. Give i) the possible outcomes of this measurement, ii) the probabilities for each of these outcomes, and iii) the state directly after the measurement. Assume for simplicity that the eigenvalues λ_i of the operator $\hat{O}$, corresponding to the eigenstates $|\psi_i\rangle$, are non-degenerate.

d) (*10 points*) Give an example of a practical application of quantum-mechanical superpositions that is currently envisioned, or is already in use.

e) (*20 points*) Sketch (as a function of B) — without calculations — the eigenvalues of the two-level hamiltonian $\hat{H} = -B|\uparrow\rangle\langle\uparrow| + t|\uparrow\rangle\langle\downarrow| + t|\downarrow\rangle\langle\uparrow|$ for the cases that $t = 0$ and that $t > 0$.

2. PARTICLE WITH SPIN $S = 3/2$ (*120 POINTS*)

In this exercise, we consider a particle with spin $S = 3/2$. Initially, we discard the orbital motion of the particle. Assume that the particle is in the state

$$|\Psi\rangle = \sum_{m_S=-S}^{m_S=S} a_{m_S}|S, m_S\rangle \ , \tag{1}$$

where the states $|S, m_S\rangle$ are simultaneous eigenstates of $\hat{S}_z$ and $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$, and $a_{-3/2}, a_{-1/2}, a_{1/2}, a_{3/2}$ are complex numbers.

a) (*10 points*) What are the possible outcomes for a measurement of the total spin, $\hat{S}^2$?

b) (*30 points*) Give the expectation value of $\hat{S}_y$ for the state in Eq. (1). (NB: Note the appendix at the end of the exam!)

Assume now that the system is described by the density matrix

$$\hat{\rho} = \sum_{m_S=-S}^{m_S=S} p_{m_S} |S, m_S\rangle\langle S, m_S| \,, \tag{2}$$

with p_{m_S} the probability to be in the state $|S, m_S\rangle$.

c) (*20 points*) Give the expectation value of $\hat{S}_x$ and $\hat{S}_z$ for a system described by the density matrix in Eq. (2)

d) (*30 points*) Consider now the case that the system is in thermal equilibrium with temperature T, and described by the hamiltonian $\hat{H} = K_1\hat{S}_x + K_2\hat{S}_x^2$, with K_1, K_2 constants. Give the expectation value of $\hat{S}_x$.

Consider now the situation that the particle also has three-dimensional orbital motion, and is described by the state $|\Psi\rangle$, which is now determined by the spinor wave function

$$\psi(\mathbf{x}) = \begin{pmatrix} \psi_{3/2}(\mathbf{x}) \\ \psi_{1/2}(\mathbf{x}) \\ \psi_{-1/2}(\mathbf{x}) \\ \psi_{-3/2}(\mathbf{x}) \end{pmatrix} \,. \tag{3}$$

of which the components are defined as $\psi_{m_S}(\mathbf{x}) = (\langle S, m_S| \langle \mathbf{x}|)|\Psi\rangle$.

e) (*10 points*) Give the expectation values of $\hat{S}_x$ and the momentum operator $\hat{\mathbf{p}}$ in terms of the components of the spinor wave function in Eq. (3).

Consider now the situation that the particle moves in three dimensions subject to the cental potential $V(\mathbf{x}) = V(|\mathbf{x}|)$, and the spin-orbit coupling $\alpha\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$. The total hamiltonian is therefore

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \alpha\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} + V(|\mathbf{x}|) \,. \tag{4}$$

f) (*20 points*) Argue that the eigenstates are labelled by the total orbital angular momentum quantum number ℓ, and also by S, and give the relative energy splitting between states for a given ℓ. Also give the degeneracies of these energy eigenstates.

3. SPIN-ORBIT COUPLING (*100 POINTS*)

Consider a two-dimensional electron gas (with periodic boundary conditions) with so-called Rashba spin-orbit interactions. The hamiltonian is given by

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \alpha\hat{S}_x\hat{p}_y \,,$$

where $\alpha > 0$ is a constant and

$$\hat{\mathbf{p}} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \end{pmatrix} \,,$$

is the two-dimensional momentum operator while $\hat{S}_x$ and $\hat{S}_y$ are spin one-half operators.

a) (*20 points*) Write the state of the system as $|\psi\rangle = |\mathbf{p}\rangle|\chi_\mathbf{p}\rangle$ with $|\mathbf{p}\rangle$ the eigenstates of the momentum operator and $|\chi_\mathbf{p}\rangle$ an element of the spin part of the Hilbert space. Show, from the time-independent Schrödinger equation, that $|\chi_\mathbf{p}\rangle$ obeys

$$\left[\frac{\mathbf{p}^2}{2m} + \alpha \hat{S}_x p_y\right]|\chi_\mathbf{p}\rangle = E|\chi_\mathbf{p}\rangle ,$$

with E the energy.

b) (*20 points*) Write $|\chi_\mathbf{p}\rangle = \chi_\uparrow(\mathbf{p})|\uparrow\rangle + \chi_\downarrow(\mathbf{p})|\downarrow\rangle$, with $|\uparrow\rangle$ and $|\downarrow\rangle$ the eigenstates of the operator describing the spin in the z-direction. Show that

$$\begin{pmatrix} \chi_\uparrow(\mathbf{p}) \\ \chi_\downarrow(\mathbf{p}) \end{pmatrix}$$

is determined by

$$\begin{pmatrix} \frac{\mathbf{p}^2}{2m} & \frac{\alpha\hbar}{2}p_y \\ \frac{\alpha\hbar}{2}p_y & \frac{\mathbf{p}^2}{2m} \end{pmatrix}\begin{pmatrix} \chi_\uparrow(\mathbf{p}) \\ \chi_\downarrow(\mathbf{p}) \end{pmatrix} = E\begin{pmatrix} \chi_\uparrow(\mathbf{p}) \\ \chi_\downarrow(\mathbf{p}) \end{pmatrix} , \tag{5}$$

c) (*10 points*) Calculate the energy eigenvalues of the hamiltonian using Eq. (5).

d) (*20 points*) Determine the corresponding eigenstates of the hamiltonian (NB: you do not need to normalize these eigenstates).

e) (*20 points*) Assume now that the system is subjected to an electromagnetic vector potential $\mathbf{A}(\hat{\mathbf{x}})$. Determine the velocity operator $\frac{d\hat{\mathbf{x}}}{dt} \equiv \frac{1}{i\hbar}[\hat{\mathbf{x}}, \hat{H}]$, with $[\cdot,\cdot]$ the commutator, and $\hat{\mathbf{x}}$ the position operator.

Appendix — Possibly, you may want to make use of these results:

- In the presence of a vector potential $\mathbf{A}$, the momentum in the hamiltonian is replaced according to the so-called minimal substitution $\hat{\mathbf{p}} \to \hat{\mathbf{p}} - q\mathbf{A}/c$.

- For $S = 3/2$ and with respect to the basis $\{|3/2, 3/2\rangle, |3/2, 1/2\rangle, |3/2, -1/2\rangle, |3/2, -3/2\rangle\}$, the spin operators have the matrix representation

$$S_x = \hbar\frac{\sqrt{3}}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{2}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad S_y = \hbar\frac{\sqrt{3}}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & \frac{-2i}{\sqrt{3}} & 0 \\ 0 & \frac{2i}{\sqrt{3}} & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \quad S_z = \hbar\begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} . \tag{6}$$

- For a system described by the density matrix $\hat{\rho}$ the expectation value of an observable $\hat{O}$ is given by $Tr[\hat{\rho}\hat{A}]$, where $Tr[\cdots]$ is the trace over Hilbert space. For a system in thermal equilibrium at temperature T, the density matrix is $\hat{\rho} = e^{-\hat{H}/(k_B T)}/Tr[e^{-\hat{H}/(k_B T)}]$.

- The basis of the combined Hilbert space of two angular momenta with values of total angular momentum j_1 and j_2, can be labelled by the quantum number j, in terms of which the eigenvalues of the total angular momentum of the combined system are $j(j+1)\hbar^2$. Then, j takes on integer values from $|j_1 - j_2|$ to $j_1 + j_2$.

- For $S = 1/2$ the spin operators are given — with respect to the ordered basis

$$\{|1/2, 1/2\rangle, |1/2, -1/2\rangle\} \equiv \{|\uparrow\rangle, |\downarrow\rangle\} ,$$

— by $S_\alpha = \hbar\tau_\alpha/2$ ($\alpha \in \{x, y, z\}$) with the Pauli matrices

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

- The spin commutation relations are $[\hat{S}_\alpha, \hat{S}_\beta] = i\hbar\epsilon_{\alpha\beta\gamma}\hat{S}_\gamma$, where α, β, γ denote Cartesian components, the summation convention is implied, and $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor.